Properties of CLS

Ethan, Tom, Jacob, Tarek

Contents

1 CLS closure

Throughout the following calculations of closures, there are red symbols underneath certain lines, indicating the symmetry properties of the term directly above. S indicates that the term is symmetric, A indicates that the term is asymmetric, and ? indicates that the term is neither symmetric nor asymmetric. In order to handle these ? terms, we group those terms that are neither symmetric nor asymmetric by field later on. For example, after line (7), we collect all terms produced by the closure of the M field that are neither symmetric nor asymmetric and include the V field, denoting these terms as "V? terms". [1]

1.1 K field

$$D_aD_bK = D_a\rho_b - D_a\zeta_b = \{\underset{A}{iC_{ab}}M + \underset{A}{(\gamma^5)_{ab}}N + \tfrac{i}{2}\underset{S}{(\gamma^\mu)_{ab}}V_\mu + \tfrac{1}{2}\underset{A}{(\gamma^5\gamma^\mu)_{ab}}U_\mu\}$$

$$-\{-i\underset{S}{(\gamma^\mu)_{ab}}(\partial_\mu K) + \underset{A}{(\gamma^5\gamma^\mu)_{ab}}(\partial_\mu L) + \tfrac{i}{2}\underset{S}{(\gamma^\mu)_{ab}}V_\mu - \tfrac{1}{2}\underset{A}{(\gamma^5\gamma^\mu)_{ab}}U_\mu\} \tag{1}$$

$$\begin{aligned}\{D_a, D_b\}K &= i(\gamma^\mu)_{ab}V_\mu + 2i(\gamma^\mu)_{ab}(\partial_\mu K) - i(\gamma^\mu)_{ab}V_\mu \\ &= 2i(\gamma^\mu)_{ab}(\partial_\mu K)\end{aligned} \tag{2}$$

1.2 L field

$$\begin{aligned}&D_aD_bL = -i(\gamma^5)_b{}^c(D_a\rho_c + D_a\zeta_c) \\ &= -i(\gamma^5)_b{}^c\{iC_{ac}M + (\gamma^5)_{ac}N + \tfrac{i}{2}(\gamma^\mu)_{ac}V_\mu + \tfrac{1}{2}(\gamma^5\gamma^\mu)_{ac}U_\mu\} \\ &-i(\gamma^5)_b{}^c\{-i(\gamma^\mu)_{ac}(\partial_\mu K) + (\gamma^5\gamma^\mu)_{ac}(\partial_\mu L) + \tfrac{i}{2}(\gamma^\mu)_{ac}V_\mu - \tfrac{1}{2}(\gamma^5\gamma^\mu)_{ac}U_\mu\} \\ &= -(\gamma^5)_{ba}M + i(\gamma^5\gamma^5)_{ba}N + \tfrac{1}{2}(\gamma^5\gamma^\mu)_{ba}V_\mu + \tfrac{i}{2}(\gamma^5\gamma^5\gamma^\mu)_{ba}U_\mu \\ &-(\gamma^5\gamma^\mu)_{ba}(\partial_\mu K) + i(\gamma^5\gamma^5\gamma^\mu)_{ba}(\partial_\mu L) + \tfrac{1}{2}(\gamma^5\gamma^\mu)_{ba}V_\mu - \tfrac{i}{2}(\gamma^5\gamma^5\gamma^\mu)_{ba}U_\mu \\ &= -\underset{A}{(\gamma^5)_{ba}}M + \underset{A}{iC_{ba}}N + \underset{A}{(\gamma^5\gamma^\mu)_{ba}}V_\mu - \underset{A}{(\gamma^5\gamma^\mu)_{ba}}(\partial_\mu K) + i\underset{S}{(\gamma^\mu)_{ba}}(\partial_\mu L)\end{aligned} \tag{3}$$

$$\{D_a, D_b\}L = 2i(\gamma^\mu)_{ab}(\partial_\mu L) \tag{4}$$

1.3 M field

$$D_aD_bM = D_a\beta_b - \tfrac{1}{2}(\gamma^\mu)_b{}^c(\partial_\mu D_a\rho_c) = X_{ab} + Y_{ab} \tag{5}$$

$$X_{ab} = -\eta^{\mu\nu}\partial_\mu\partial_\nu\left(\underset{A}{iC_{ab}}K + \underset{A}{(\gamma^5)_{ab}}L\right) + \tfrac{i}{2}\underset{S}{(\gamma^\mu)_{ab}}(\partial_\mu M) + \tfrac{1}{2}\underset{A}{(\gamma^5\gamma^\mu)_{ab}}(\partial_\mu N)$$

$$+\tfrac{i}{2}\underset{?}{(\gamma^\mu\gamma^\nu)_{ab}}(\partial_\mu V_\nu) + \tfrac{i}{4}\underset{?}{(\gamma^\nu\gamma^\mu)_{ab}}(\partial_\mu V_\nu) + \tfrac{1}{2}\underset{?}{(\gamma^5\gamma^\mu\gamma^\nu)_{ab}}(\partial_\mu U_\nu) + \tfrac{1}{4}\underset{?}{(\gamma^5\gamma^\nu\gamma^\mu)_{ab}}(\partial_\mu U_\nu) \tag{6}$$

$$Y_{ab} = -\tfrac{1}{2}\left(\gamma^\mu\right)_b{}^c \partial_\mu \{iC_{ac}M + \left(\gamma^5\right)_{ac} N + \tfrac{i}{2}\left(\gamma^\nu\right)_{ac} V_\nu + \tfrac{1}{2}\left(\gamma^5\gamma^\nu\right)_{ac} U_\nu\}$$
$$= \tfrac{i}{2}\left(\gamma^\mu\right)_{ba}\underset{S}{(\partial_\mu M)} + \tfrac{1}{2}\left(\gamma^\mu\gamma^5\right)_{ba}\underset{A}{(\partial_\mu N)} - \tfrac{i}{4}\left(\gamma^\mu\gamma^\nu\right)_{ba}\underset{?}{(\partial_\mu V_\nu)} + \tfrac{1}{4}\left(\gamma^\mu\gamma^5\gamma^\nu\right)_{ba}\underset{?}{(\partial_\mu U_\nu)} \tag{7}$$

V? terms:

$$\{\tfrac{i}{2}(\gamma^\mu\gamma^\nu)_{ab} + \tfrac{i}{4}(\gamma^\nu\gamma^\mu)_{ab} - \tfrac{i}{4}(\gamma^\mu\gamma^\nu)_{ba}\}(\partial_\mu V_\nu) \tag{8}$$

Aside:

$$(\gamma^\mu\gamma^\nu)_{ba} = (\gamma^\mu)_b{}^c(\gamma^\nu)_{ca}$$
$$= (\gamma^\nu)_{ac}(\gamma^\mu)_b{}^c = -(\gamma^\nu)_a{}^c(\gamma^\mu)_{bc} = -(\gamma^\nu\gamma^\mu)_{ab} \tag{9}$$

End of aside

$$\left\{\tfrac{i}{2}\left(\gamma^\mu\gamma^\nu\right)_{ab} + \tfrac{i}{4}\left(\gamma^\nu\gamma^\mu\right)_{ab} + \tfrac{i}{4}\left(\gamma^\nu\gamma^\mu\right)_{ab}\right\}(\partial_\mu V_\nu)$$
$$= \tfrac{i}{2}\left\{2\eta^{\mu\nu}(\mathbf{I})_{ab}\right\}(\partial_\mu V_\nu)$$
$$= i\eta^{\mu\nu}\underset{A}{C_{ab}}(\partial_\mu V_\nu) \tag{10}$$

U? terms:

$$\left\{\tfrac{1}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ab} + \tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ab} - \tfrac{1}{4}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}\right\}(\partial_\mu U_\nu) \tag{11}$$

Aside:

$$\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba} = \left(\gamma^5\right)_b{}^c\left(\gamma^\mu\gamma^\nu\right)_{ca} = -\left(\gamma^\nu\gamma^\mu\right)_{ac}\left(\gamma^5\right)_b{}^c = \left(\gamma^\nu\gamma^\mu\right)_a{}^c\left(\gamma^5\right)_{bc} = -\left(\gamma^\nu\gamma^\mu\gamma^5\right)_{ab}$$
$$= -\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ab} \tag{12}$$

End of aside

$$\left\{\tfrac{1}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ab} + \tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ab} + \tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ab}\right\}(\partial_\mu U_\nu)$$
$$= \tfrac{1}{2}\{\gamma^5(2\eta^{\mu\nu}\mathbf{I})\}_{ab}(\partial_\mu V_\nu)$$
$$= \eta^{\mu\nu}\underset{A}{\left(\gamma^5\right)_{ab}}(\partial_\mu U_\nu) \tag{13}$$

$$\{D_a, D_b\}M = 2i(\gamma^\mu)_{ab}(\partial_\mu M) \tag{14}$$

1.4 N field

$$D_a D_b N = -i\left(\gamma^5\right)_b{}^c(D_a\beta_c) + \tfrac{i}{2}\left(\gamma^5\gamma^\mu\right)_b{}^c(\partial_\mu D_a\rho_c) = X_{ab} + Y_{ab} \tag{15}$$

$$
\begin{aligned}
X_{ab} &= -i\left(\gamma^5\right)_b{}^c\{-\eta^{\mu\nu}\partial_\mu\partial_\nu\left(iC_{ac}K+\left(\gamma^5\right)_{ac}L\right)+\tfrac{i}{2}\left(\gamma^\mu\right)_{ac}\left(\partial_\mu M\right)+\tfrac{1}{2}\left(\gamma^5\gamma^\mu\right)_{ac}\left(\partial_\mu N\right)\\
&+\tfrac{i}{2}\left(\gamma^\mu\gamma^\nu\right)_{ac}\left(\partial_\mu V_\nu\right)+\tfrac{i}{4}\left(\gamma^\nu\gamma^\mu\right)_{ac}\left(\partial_\mu V_\nu\right)+\tfrac{1}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ac}\left(\partial_\mu U_\nu\right)+\tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ac}\left(\partial_\mu U_\nu\right)\}\\
&= \eta^{\mu\nu}\left(\gamma^5\right)_{ba}\left(\partial_\mu\partial_\nu K\right)-i\eta^{\mu\nu}\left(\gamma^5\gamma^5\right)_{ba}\left(\partial_\mu\partial_\nu L\right)+\tfrac{1}{2}\left(\gamma^5\gamma^\mu\right)_{ba}\left(\partial_\mu M\right)+\tfrac{i}{2}\left(\gamma^5\gamma^5\gamma^\mu\right)_{ba}\left(\partial_\mu N\right)\\
&-\tfrac{1}{2}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ba}\left(\partial_\mu V_\nu\right)-\tfrac{1}{4}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}\left(\partial_\mu V_\nu\right)+\tfrac{i}{2}\left(\gamma^5\gamma^5\gamma^\nu\gamma^\mu\right)_{ba}\left(\partial_\mu U_\nu\right)+\tfrac{i}{4}\left(\gamma^5\gamma^5\gamma^\mu\gamma^\nu\right)_{b_a}\left(\partial_\mu U_\nu\right)\\
&= \eta^{\mu\nu}\underset{A}{\left(\gamma^5\right)_{ba}}\left(\partial_\mu\partial_\nu K\right)-i\eta^{\mu\nu}\underset{A}{C_{ba}}\left(\partial_\mu\partial_\nu L\right)+\tfrac{1}{2}\underset{A}{\left(\gamma^5\gamma^\mu\right)_{ba}}\left(\partial_\mu M\right)+\tfrac{i}{2}\underset{S}{\left(\gamma^\mu\right)_{ba}}\left(\partial_\mu N\right)\\
&-\tfrac{1}{2}\underset{?}{\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ba}}\left(\partial_\mu V_\nu\right)-\tfrac{1}{4}\underset{?}{\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}}\left(\partial_\mu V_\nu\right)+\tfrac{i}{2}\underset{?}{\left(\gamma^\nu\gamma^\mu\right)_{ba}}\left(\partial_\mu U_\nu\right)+\tfrac{i}{4}\underset{?}{\left(\gamma^\mu\gamma^\nu\right)_{ba}}\left(\partial_\mu U_\nu\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
Y_{ab} &= \tfrac{i}{2}\left(\gamma^5\gamma^\mu\right)_b{}^c\partial_\mu\left\{iC_{ac}M+\left(\gamma^5\right)_{ac}N+\tfrac{i}{2}\left(\gamma^\nu\right)_{ac}V_\nu+\tfrac{1}{2}\left(\gamma^5\gamma^\nu\right)_{ac}U_\nu\right\}\\
&= \tfrac{1}{2}\left(\gamma^5\gamma^\mu\right)_{ba}\left(\partial_\mu M\right)-\tfrac{i}{2}\left(\gamma^5\gamma^\mu\gamma^5\right)_{ba}\left(\partial_\mu N\right)-\tfrac{1}{4}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}\left(\partial_\mu V_\nu\right)-\tfrac{i}{4}\left(\gamma^5\gamma^\mu\gamma^5\gamma^\nu\right)_{ba}\left(\partial_\mu U_\nu\right)\\
&= \tfrac{1}{2}\underset{A}{\left(\gamma^5\gamma^\mu\right)_{ba}}\left(\partial_\mu M\right)+\tfrac{i}{2}\underset{S}{\left(\gamma^\mu\right)_{ba}}\left(\partial_\mu N\right)-\tfrac{1}{4}\underset{?}{\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}}\left(\partial_\mu V_\nu\right)+\tfrac{i}{4}\underset{?}{\left(\gamma^\mu\gamma^\nu\right)_{ba}}\left(\partial_\mu U_\nu\right)
\end{aligned}
\tag{17}
$$

V? terms:

$$
\begin{aligned}
&-\tfrac{1}{2}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ba}\left(\partial_\mu V_\nu\right)-\tfrac{1}{4}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}\left(\partial_\mu V_\nu\right)_{ba}-\tfrac{1}{4}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ba}\left(\partial_\mu V_\nu\right)\\
&= -\tfrac{1}{2}\left\{\gamma^5\left(2\eta^{\mu\nu}\mathbf{I}\right)\right\}_{ba}\left(\partial_\mu V_\nu\right) = -\eta^{\mu\nu}\underset{A}{\left(\gamma^5\right)_{ba}}\left(\partial_\mu V_\nu\right)
\end{aligned}
\tag{18}
$$

U? terms:

$$
\begin{aligned}
&\tfrac{i}{2}(\gamma^\nu\gamma^\mu)_{ba}(\partial_\mu U_\nu)+\tfrac{i}{4}(\gamma^\mu\gamma^\nu)_{ba}(\partial_\mu U_\nu)+\tfrac{i}{4}(\gamma^\mu\gamma^\nu)_{ba}(\partial_\mu U_\nu)\\
&= \tfrac{i}{2}\{2\eta^{\mu\nu}\mathbf{I}\}_{ba}(\partial_\mu U_\nu)\\
&= i\eta^{\mu\nu}\underset{A}{C_{ba}}(\partial_\mu U_\nu)
\end{aligned}
\tag{19}
$$

$$
\left\{D_a, D_b\right\}N = 2i\left(\gamma^\mu\right)_{ab}\left(\partial_\mu N\right) \tag{20}
$$

1.5 V_μ field

$$
\begin{aligned}
D_aD_bV_\mu &= -\left(\gamma_\mu\right)_b{}^c\left(D_a\beta_c\right)-\left(\gamma_\mu\gamma^\nu\right)_b{}^c\left(\partial_\nu D_a\zeta_c\right)+\left(\partial_\mu D_a\rho_b\right)+\tfrac{1}{2}\left(\gamma^\nu\gamma_\mu\right)_b{}^c\left(\partial_\nu D_a\rho_c\right)\\
&= X_{ab\mu}+Y_{ab\mu}+Z_{ab\mu}+W_{ab\mu}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
X_{ab\mu} &= -\left(\gamma_\mu\right)_b{}^c\{-\eta^{\lambda\nu}\partial_\lambda\partial_\nu\left(iC_{ac}K + \left(\gamma^5\right)_{ac}L\right) \\
&+\tfrac{i}{2}\left(\gamma^\lambda\right)_{ac}\left(\partial_\lambda M\right) + \tfrac{1}{2}\left(\gamma^5\gamma^\lambda\right)_{ac}\left(\partial_\lambda N\right) \\
&+\tfrac{i}{2}\left(\gamma^\lambda\gamma^\nu\right)_{ac}\left(\partial_\lambda V_\nu\right) + \tfrac{i}{4}\left(\gamma^\nu\gamma^\lambda\right)_{ac}\left(\partial_\lambda V_\nu\right) + \tfrac{1}{2}\left(\gamma^5\gamma^\lambda\gamma^\nu\right)_{ac}\left(\partial_\lambda U_\nu\right) + \tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\lambda\right)_{ac}\left(\partial_\lambda U_\nu\right)\} \\
&= -i\eta^{\lambda\nu}\underset{S}{\left(\gamma_\mu\right)_{ba}}\left(\partial_\lambda\partial_\nu K\right) - \eta^{\lambda\nu}\underset{A}{\left(\gamma_\mu\gamma^5\right)_{ba}}\left(\partial_\lambda\partial_\nu L\right) - \tfrac{i}{2}\underset{?}{\left(\gamma_\mu\gamma^\lambda\right)_{ba}}\left(\partial_\lambda M\right) + \tfrac{1}{2}\underset{?}{\left(\gamma_\mu\gamma^5\gamma^\lambda\right)_{ba}}\left(\partial_\lambda N\right) \\
&+\tfrac{i}{2}\underset{?}{\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}}\left(\partial_\lambda V_\nu\right) + \tfrac{i}{4}\underset{?}{\left(\gamma_\mu\gamma^\lambda\gamma^\nu\right)_{ba}}\left(\partial_\lambda V_\nu\right) + \tfrac{1}{2}\underset{?}{\left(\gamma_\mu\gamma^5\gamma^\nu\gamma^\lambda\right)_{ba}}\left(\partial_\lambda U_\nu\right) + \tfrac{1}{4}\underset{?}{\left(\gamma_\mu\gamma^5\gamma^\lambda\gamma^\nu\right)_{ba}}\left(\partial_\lambda U_\nu\right)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
Y_{ab\mu} &= -\left(\gamma_\mu\gamma^\nu\right)_b{}^c\partial_\nu\left\{-i\left(\gamma^\lambda\right)_{ac}\left(\partial_\lambda K\right) + \left(\gamma^5\gamma^\lambda\right)_{ac}\left(\partial_\lambda L\right) + \tfrac{i}{2}\left(\gamma^\lambda\right)_{ac}V_\lambda - \tfrac{1}{2}\left(\gamma^5\gamma^\lambda\right)_{ac}U_\lambda\right\} \\
&= i\underset{?}{\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}}\left(\partial_\nu\partial_\lambda K\right) + \underset{?}{\left(\gamma_\mu\gamma^\nu\gamma^5\gamma^\lambda\right)_{ba}}\left(\partial_\nu\partial_\lambda L\right) - \tfrac{i}{2}\underset{?}{\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}}\left(\partial_\nu V_\lambda\right) - \tfrac{1}{2}\underset{?}{\left(\gamma_\mu\gamma^\nu\gamma^5\gamma^\lambda\right)_{ba}}\left(\partial_\nu U_\lambda\right)
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
Z_{ab\mu} &= \partial_\mu\left\{iC_{ab}M + \left(\gamma^5\right)_{ab}N + \tfrac{i}{2}\left(\gamma^\nu\right)_{ab}V_\nu + \tfrac{1}{2}\left(\gamma^5\gamma^\nu\right)_{ab}U_\nu\right\} \\
&= i\underset{A}{C_{ab}}(\partial_\mu M) + \underset{A}{(\gamma^5)_{ab}}(\partial_\mu N) + \tfrac{i}{2}\underset{S}{(\gamma^\nu)_{ab}}(\partial_\mu V_\nu) + \tfrac{1}{2}\underset{A}{(\gamma^5\gamma^\nu)_{ab}}(\partial_\mu U_\nu)
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
W_{ab\mu} &= \tfrac{1}{2}\left(\gamma^\nu\gamma_\mu\right)_b{}^c\partial_\nu\left\{iC_{ac}M + \left(\gamma^5\right)_{ac}N + \tfrac{i}{2}\left(\gamma^\lambda\right)_{ac}V_\lambda + \tfrac{1}{2}\left(\gamma^5\gamma^\lambda\right)_{ac}U_\lambda\right\} \\
&= -\tfrac{i}{2}\underset{?}{\left(\gamma^\nu\gamma_\mu\right)_{ba}}\left(\partial_\nu M\right) - \tfrac{1}{2}\underset{?}{\left(\gamma^\nu\gamma_\mu\gamma^5\right)_{ba}}\left(\partial_\nu N\right) + \tfrac{i}{4}\underset{?}{\left(\gamma^\nu\gamma_\mu\gamma^\lambda\right)_{ba}}\left(\partial_\nu V_\lambda\right) - \tfrac{1}{4}\underset{?}{\left(\gamma^\nu\gamma_\mu\gamma^5\gamma^\lambda\right)_{ba}}\left(\partial_\nu U_\lambda\right)
\end{aligned}
\tag{25}
$$

M? Terms:

$$
\begin{aligned}
&-\tfrac{i}{2}\left(\gamma_\mu\gamma^\lambda\right)_{ba}\left(\partial_\lambda M\right) - \tfrac{i}{2}\left(\gamma^\nu\gamma_\mu\right)_{ba}\left(\partial_\nu M\right) \\
&= -\tfrac{i}{2}\left\{\gamma_\mu\gamma^\nu + \gamma^\nu\gamma_\mu\right\}_{ba}\left(\partial_\nu M\right) \\
&= -\tfrac{i}{2}\eta_{\mu\lambda}\left\{\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\right\}_{ba}\left(\partial_\nu M\right) = -\tfrac{i}{2}\eta_{\mu\lambda}\{2\eta^{\nu\lambda}\mathbf{I}\}_{ba}\left(\partial_\nu M\right) \\
&= -i\delta_\mu{}^\nu\underset{A}{C_{ba}}\left(\partial_\nu M\right)
\end{aligned}
\tag{26}
$$

N? Terms:

$$
\tfrac{1}{2}\left(\gamma_\mu\gamma^5\gamma^\lambda\right)_{ba}\left(\partial_\lambda N\right) - \tfrac{1}{2}\left(\gamma^\nu\gamma_\mu\gamma^5\right)_{ba}\left(\partial_\nu N\right) = -\tfrac{1}{2}\left\{\gamma^5\gamma_\mu\gamma^\nu + \gamma^5\gamma^\nu\gamma_\mu\right\}_{ba}\left(\partial_\nu N\right)
\tag{27}
$$

Aside:

$$\gamma_\mu\gamma^\nu + \gamma^\nu\gamma_\mu = \eta_{\mu\lambda}\left\{\gamma^\lambda, \gamma^\nu\right\} = 2\delta_\mu{}^\nu\mathbf{I} \tag{28}$$

End of aside

$$= -\tfrac{1}{2}\{\underset{A}{\gamma^5(2\delta_\mu{}^\nu\mathbf{I})}\}_{ba}(\partial_\nu N) = -\delta_\mu{}^\nu(\gamma^5)_{ba}(\partial_\nu N) \tag{29}$$

V? terms:

$$\begin{aligned}
&\tfrac{i}{2}\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}(\partial_\lambda V_\nu) + \tfrac{i}{4}\left(\gamma_\mu\gamma^\lambda\gamma^\nu\right)_{ba}(\partial_\lambda V_\nu) - \tfrac{i}{2}\left(\gamma_\mu\gamma^\lambda\gamma^\nu\right)_{ba}(\partial_\lambda V_\nu) + \tfrac{i}{4}\left(\gamma^\lambda\gamma_\mu\gamma^\nu\right)_{ba}(\partial_\lambda V_\nu)\\
&= \tfrac{i}{2}\left\{\gamma_\mu\left[\gamma^\nu, \gamma^\lambda\right]\right\}_{ba}(\partial_\lambda V_\nu) + \tfrac{i}{4}\left\{\left(\gamma_\mu\gamma^\lambda + \gamma^\lambda\gamma_\mu\right)\gamma^\nu\right\}_{ba}(\partial_\lambda V_\nu)\\
&= \left(\gamma_\mu\sigma^{\nu\lambda}\right)_{ba}(\partial_\lambda V_\nu) + \tfrac{i}{4}\left\{2\delta_\mu{}^\lambda\gamma^\nu\right\}_{ba}(\partial_\lambda V_\nu)\\
&= \tfrac{i}{2}\eta_{\mu\rho}(2i\epsilon^{\varphi\rho\nu\lambda}\gamma_\varphi\gamma^5 + 2\eta^{\rho[\nu}\gamma^{\lambda]})_{ba}(\partial_\lambda V_\nu) + \tfrac{i}{2}(\gamma^\nu)_{ba}(\partial_\mu V_\nu)\\
&= -\eta_{\mu\rho}\epsilon^{\varphi\rho\nu\lambda}(\gamma_\varphi\gamma^5)_{ba}(\partial_\lambda V_\nu) + i\delta_\mu{}^\nu(\partial_\lambda V_\nu) - i\delta_\mu{}^\lambda(\gamma^\nu)_{ba}(\partial_\lambda V_\nu) + \tfrac{i}{2}(\gamma^\nu)_{ba}(\partial_\mu V_\nu)\\
&= -\eta_{\mu\rho}\epsilon^{\varphi\rho\nu\lambda}\underset{A}{(\gamma_\varphi\gamma^5)_{ba}}(\partial_\lambda V_\nu) + i\underset{S}{(\gamma^\lambda)_{ba}}(\partial_\lambda V_\mu) - \tfrac{i}{2}\underset{S}{(\gamma^\nu)_{ba}}(\partial_\mu V_\nu)
\end{aligned} \tag{30}$$

U? terms:

$$\begin{aligned}
&-\tfrac{1}{2}(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\lambda U_\nu) - \tfrac{1}{4}(\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda U_\nu)\\
&-\tfrac{1}{2}(\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda U_\nu) - \tfrac{1}{4}(\gamma^5\gamma^\lambda\gamma_\mu\gamma^\nu)_{ba}(\partial_\lambda U_\nu)\\
&= -\tfrac{1}{2}\{\gamma^5\gamma_\mu(\gamma^\nu\gamma^\lambda + \gamma^\lambda\gamma^\nu)\}_{ba}(\partial_\lambda U_\nu) - \tfrac{1}{4}\{\gamma^5(\gamma_\mu\gamma^\lambda + \gamma^\lambda\gamma_\mu)\gamma^\nu\}_{ba}(\partial_\lambda U_\nu)\\
&= -\tfrac{1}{2}\{2\eta^{\nu\lambda}\gamma^5\gamma_\mu\}_{ba}(\partial_\lambda U_\nu) - \tfrac{1}{4}\{2\delta_\mu{}^\lambda\gamma^5\gamma^\nu\}_{ba}(\partial_\lambda U_\nu)\\
&= -\eta^{\nu\lambda}\underset{A}{(\gamma^5\gamma_\mu)_{ba}}(\partial_\lambda U_\nu) - \tfrac{1}{2}\underset{A}{(\gamma^5\gamma^\nu)_{ba}}(\partial_\mu U_\nu)
\end{aligned} \tag{31}$$

K? terms:

$$\begin{aligned}
&i(\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\nu\partial_\lambda K) = i(\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda\partial_\nu K) = \tfrac{i}{2}\{\gamma_\mu(\gamma^\nu\gamma^\lambda + \gamma^\lambda\gamma^\nu)\}_{ba}(\partial_\nu\partial_\lambda K)\\
&= i\eta^{\nu\lambda}\underset{S}{(\gamma_\mu)_{ba}}(\partial_\nu\partial_\lambda K)
\end{aligned} \tag{32}$$

L? terms:

$$\begin{aligned}
&\left(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}(\partial_\nu\partial_\lambda L) = (\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\nu\partial_\lambda L) = \tfrac{1}{2}\left\{\gamma^5\gamma_\mu\left(\gamma^\nu\gamma^\lambda + \gamma^\lambda\gamma^\nu\right)\right\}_{ba}(\partial_\nu\partial_\lambda L)\\
&= \eta^{\nu\lambda}\underset{A}{\left(\gamma^5\gamma_\mu\right)_{ba}}(\partial_\nu\partial_\lambda L)
\end{aligned} \tag{33}$$

$$\{D_a, D_b\}V_\mu = -2i\eta^{\lambda\nu}(\gamma_\mu)_{ba}(\partial_\lambda\partial_\nu K) + 2i\eta^{\nu\lambda}(\gamma_\mu)_{ba}(\partial_\nu\partial_\lambda K) + i(\gamma^\nu)_{ab}(\partial_\mu V_\nu)$$
$$+2i(\gamma^\lambda)_{ba}(\partial_\lambda V_\mu) - i(\gamma^\nu)_{ba}(\partial_\mu V_\nu)$$
$$= 2i(\gamma^\lambda)_{ba}(\partial_\lambda V_\mu) \tag{34}$$

1.6 U_μ field

$$D_a D_b U_\mu = i\left(\gamma^5\gamma_\mu\right)_b{}^c(D_a\beta_c) - i\left(\gamma^5\gamma_\mu\gamma^\nu\right)_b{}^c(\partial_\nu D_a\zeta_c) - i\left(\gamma^5\right)_b{}^c(\partial_\mu D_a\rho_c) - \tfrac{i}{2}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_b{}^c(\partial_\nu D_a\rho_c)$$
$$= X_{ab\mu} + Y_{ab\mu} + Z_{ab\mu} + W_{ab\mu} \tag{35}$$

$$X_{ab\mu} = i\left(\gamma^5\gamma_\mu\right)_b{}^c\{-\eta^{\lambda\nu}\partial_\lambda\partial_\nu\left(iC_{ac}K + \left(\gamma^5\right)_{ac}L\right) + \tfrac{i}{2}\left(\gamma^\lambda\right)_{ac}(\partial_\lambda M)$$
$$+\tfrac{1}{2}\left(\gamma^5\gamma^\lambda\right)_{ac}(\partial_\lambda N) + \tfrac{i}{2}\left(\gamma^\lambda\gamma^\nu\right)_{ac}(\partial_\lambda V_\nu) + \tfrac{i}{4}\left(\gamma^\nu\gamma^\lambda\right)_{ac}(\partial_\lambda V_\nu)$$
$$+\tfrac{1}{2}\left(\gamma^5\gamma^\lambda\gamma^\nu\right)_{ac}(\partial_\lambda U_\nu) + \tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\lambda\right)_{ac}(\partial_\lambda U_\nu)\}$$
$$= -\eta^{\lambda\nu}(\gamma^5\gamma_\mu)_{ba}(\partial_\lambda\partial_\nu K) + i\eta^{\lambda\nu}(\gamma^5\gamma_\mu\gamma^5)_{ba}(\partial_\lambda\partial_\nu L) - \tfrac{1}{2}(\gamma^5\gamma_\mu\gamma^\lambda)_{ba}(\partial_\lambda M)$$
$$-\tfrac{i}{2}(\gamma^5\gamma_\mu\gamma^5\gamma^\lambda)_{ba}(\partial_\lambda N) + \tfrac{1}{2}(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\lambda V_\nu)$$
$$+\tfrac{1}{4}(\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda V_\nu) - \tfrac{i}{2}(\gamma^5\gamma_\mu\gamma^5\gamma^\nu\gamma^\lambda)_{ba}(\partial_\lambda U_\nu) - \tfrac{i}{4}(\gamma^5\gamma_\mu\gamma^5\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda U_\nu)$$
$$= -\eta^{\lambda\nu}\underset{A}{(\gamma^5\gamma_\mu)_{ba}}(\partial_\lambda\partial_\nu K) - i\eta^{\lambda\nu}\underset{S}{(\gamma_\mu)_{ba}}(\partial_\lambda\partial_\nu L)$$
$$-\tfrac{1}{2}\underset{?}{(\gamma^5\gamma_\mu\gamma^\lambda)_{ba}}(\partial_\lambda M) + \tfrac{i}{2}\underset{?}{(\gamma_\mu\gamma^\lambda)_{ba}}(\partial_\lambda N)$$
$$+\tfrac{1}{2}\underset{?}{(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}}(\partial_\lambda V_\nu) + \tfrac{1}{4}\underset{?}{(\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}}(\partial_\lambda V_\nu)$$
$$+\tfrac{i}{2}\underset{?}{(\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}}(\partial_\lambda U_\nu) + \tfrac{i}{4}\underset{?}{(\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}}(\partial_\lambda U_\nu) \tag{36}$$

$$Y_{ab\mu} = -i\left(\gamma^5\gamma_\mu\gamma^\nu\right)_b{}^c\partial_\nu\left\{-i\left(\gamma^\lambda\right)_{ac}(\partial_\lambda K) + \left(\gamma^5\gamma^\lambda\right)_{ac}(\partial_\lambda L) + \tfrac{i}{2}\left(\gamma^\lambda\right)_{ac}V_\lambda - \tfrac{1}{2}\left(\gamma^5\gamma^\lambda\right)_{ac}U_\lambda\right\}$$

$$= -(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\nu\partial_\lambda K) + i(\gamma^5\gamma_\mu\gamma^\nu\gamma^5\gamma^\lambda)_{ba}(\partial_\nu\partial_\lambda L) + \tfrac{1}{2}(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\nu V_\lambda) - \tfrac{i}{2}((\gamma^5\gamma_\mu\gamma^\nu\gamma^5\gamma^\lambda)_{ba}(\partial_\nu U_\lambda)$$

$$= -\eta^{\nu\lambda}\left(\gamma^5\gamma_\mu\right)_{ba}(\partial_\nu\partial_\lambda K) + i\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}(\partial_\nu\partial_\lambda L) + \tfrac{1}{2}\left(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}(\partial_\nu V_\lambda) - \tfrac{i}{2}\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}(\partial_\nu U_\lambda)$$

$$= -\eta^{\nu\lambda}\underset{A}{\left(\gamma^5\gamma_\mu\right)_{ba}}(\partial_\nu\partial_\lambda K) + i\eta^{\nu\lambda}\underset{S}{(\gamma_\mu)_{ba}}(\partial_\nu\partial_\lambda L)$$

$$+\tfrac{1}{2}\underset{?}{\left(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}}(\partial_\nu V_\lambda) - \tfrac{i}{2}\underset{?}{\left(\gamma_\mu\gamma^\nu\gamma^\lambda\right)_{ba}}(\partial_\nu U_\lambda) \tag{37}$$

$$Z_{ab\mu} = -i\left(\gamma^5\right)_b{}^c\partial_\mu\left\{iC_{ac}M + \left(\gamma^5\right)_{ac}N + \tfrac{i}{2}\left(\gamma^\nu\right)_{ac}V_\nu + \tfrac{1}{2}\left(\gamma^5\gamma^\nu\right)_{ac}U_\nu\right\}$$

$$= -\left(\gamma^5\right)_{ba}(\partial_\mu M) + i\left(\gamma^5\gamma^5\right)_{ba}(\partial_\mu N) + \tfrac{1}{2}\left(\gamma^5\gamma^\nu\right)_{ba}(\partial_\mu V_\nu) + \tfrac{i}{2}\left(\gamma^5\gamma^5\gamma^\nu\right)_{ba}(\partial_\mu U_\nu)$$

$$= -\underset{A}{\left(\gamma^5\right)_{ba}}(\partial_\mu M) + i\underset{A}{C_{ba}}(\partial_\mu N) + \tfrac{1}{2}\underset{A}{\left(\gamma^5\gamma^\nu\right)_{ba}}(\partial_\mu V_\nu) + \tfrac{i}{2}\underset{S}{\left(\gamma^\nu\right)_{ba}}(\partial_\mu U_\nu) \tag{38}$$

$$W_{ab\mu} = -\tfrac{i}{2}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_b{}^c\partial_\nu\left\{iC_{ac}M + \left(\gamma^5\right)_{ac}N + \tfrac{i}{2}\left(\gamma^\lambda\right)_{ac}V_\lambda + \tfrac{1}{2}\left(\gamma^5\gamma^\lambda\right)_{ac}U_\lambda\right\}$$

$$= -\tfrac{1}{2}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_{ba}(\partial_\nu M) + \tfrac{i}{2}\left(\gamma^5\gamma^\nu\gamma_\mu\gamma^5\right)_{ba}(\partial_\nu N) + \tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma_\mu\gamma^\lambda\right)_{ba}(\partial_\nu V_\lambda) + \tfrac{i}{4}\left(\gamma^5\gamma^\nu\gamma_\mu\gamma^5\gamma^\lambda\right)_{ba}(\partial_\nu U_\lambda)$$

$$= -\tfrac{i}{2}\underset{?}{(\gamma^5\gamma^\nu\gamma_\mu)_{ba}}(\partial_\nu M) + \tfrac{i}{2}\underset{?}{(\gamma^\nu\gamma_\mu)_{ba}}(\partial_\nu N) + \tfrac{1}{4}\underset{?}{(\gamma^5\gamma^\nu\gamma_\mu\gamma^\lambda)_{ba}}(\partial_\nu V_\lambda) + \tfrac{i}{4}\underset{?}{(\gamma^\nu\gamma_\mu\gamma^\lambda)_{ba}}(\partial_\nu U_\lambda) \tag{39}$$

M? terms:

$$-\tfrac{1}{2}\left(\gamma^5\gamma_\mu\gamma^\nu\right)_{ba}(\partial_\nu M) - \tfrac{1}{2}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_{ba}(\partial_\nu M) = -\tfrac{1}{2}\left(2\delta_\mu{}^\nu\gamma^5\right)_{ba}(\partial_\nu M)$$

$$= -\delta_\mu{}^\nu\underset{A}{\left(\gamma^5\right)_{ba}}(\partial_\nu M) \tag{40}$$

N? terms:

$$\tfrac{i}{2}\left(\gamma_\mu\gamma^\nu\right)_{ba}(\partial_\nu N) + \tfrac{i}{2}\left(\gamma^\nu\gamma_\mu\right)_{ba}(\partial_\nu N) = \tfrac{i}{2}\left(2\delta_\mu{}^\nu\mathbf{I}\right)_{ba}(\partial_\nu N) = i\delta_\mu{}^\nu\underset{A}{C_{ba}}(\partial_\nu N) \tag{41}$$

V? terms:

$$\frac{1}{2}(\gamma^5\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\lambda V_\nu) + \frac{1}{4}(\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda V_\nu) + \frac{1}{2}(\gamma^5\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda V_\nu)$$
$$+\frac{1}{4}(\gamma^5\gamma^\lambda\gamma_\mu\gamma^\nu)_{ba}(\partial_\lambda V_\nu)$$
$$= \frac{1}{2}(2\eta^{\nu\lambda}\gamma^5\gamma_\mu)_{ba}(\partial_\lambda V_\nu) + \frac{1}{4}(2\delta_\mu{}^\lambda\gamma^5\gamma^\nu)_{ba}(\partial_\lambda V_\nu) \tag{42}$$
$$= \eta^{\nu\lambda}\underset{A}{(\gamma^5\gamma_\mu)_{ba}}(\partial_\lambda V_\nu) + \frac{1}{2}\underset{A}{(\gamma^5\gamma^\nu)_{ba}}(\partial_\mu V_\nu)$$

U? terms:

$$\frac{i}{2}(\gamma_\mu\gamma^\nu\gamma^\lambda)_{ba}(\partial_\lambda U_\nu) + \frac{i}{4}(\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda U_\nu) - \frac{i}{2}(\gamma_\mu\gamma^\lambda\gamma^\nu)_{ba}(\partial_\lambda U_\nu) + \frac{i}{4}(\gamma^\lambda\gamma_\mu\gamma^\nu)_{ba}(\partial_\lambda U_\nu)$$
$$= \frac{i}{2}\left(\gamma_\mu\left[\gamma^\nu,\gamma^\lambda\right]\right)_{ba}(\partial_\lambda U_\nu) + \frac{i}{4}\left(2\delta_\mu{}^\lambda\gamma^\nu\right)_{ba}(\partial_\lambda U_\nu)$$
$$= \frac{i}{2}\eta_{\mu\rho}\left(2i\epsilon^{\varphi\rho\nu\lambda}\gamma_\varphi\gamma^5 + 2\eta^{\rho[\nu}\gamma^{\lambda]}\right)_{ba}(\partial_\lambda U_\nu) + \frac{i}{2}(\gamma^\nu)_{ba}(\partial_\mu U_\nu)$$
$$= -\eta_{\mu\rho}\epsilon^{\varphi\rho\nu\lambda}(\gamma_\varphi\gamma^5)_{ba}(\partial_\lambda U_\nu) + i\delta_\mu{}^\nu(\gamma^\lambda)_{ba}(\partial_\lambda U_\nu) - i\delta_\mu{}^\lambda(\gamma^\nu)_{ba}(\partial_\lambda U_\nu) + \frac{i}{2}(\gamma^\nu)_{ba}(\partial_\mu U_\nu)$$
$$= -\eta_{\mu\rho}\epsilon^{\varphi\rho\nu\lambda}\underset{A}{\left(\gamma_\varphi\gamma^5\right)_{ba}}(\partial_\lambda U_\nu) + i\underset{S}{\left(\gamma^\lambda\right)_{ba}}(\partial_\lambda U_\mu) - \frac{i}{2}\underset{S}{(\gamma^\nu)_{ba}}(\partial_\mu U_\nu) \tag{43}$$

$$\{D_a, D_b\}U_\mu = -2i\eta^{\lambda\nu}(\gamma_\mu)_{ba}(\partial_\lambda\partial_\nu L) + 2i\eta^{\nu\lambda}(\gamma_\mu)_{ba}(\partial_\lambda\partial_\nu L)$$
$$+i(\gamma^\nu)_{ba}(\partial_\mu U_\nu) + 2i(\gamma^\lambda)_{ba}(\partial_\lambda U_\mu) - i(\gamma^\nu)_{ba}(\partial_\mu U_\nu) \tag{44}$$
$$= 2i(\gamma^\lambda)_{ba}(\partial_\lambda U_\mu)$$

1.7 ζ_c field

$$D_a D_b\zeta_c = -i(\gamma^\mu)_{bc}(\partial_\mu D_a K) + \left(\gamma^5\gamma^\mu\right)_{bc}(\partial_\mu D_a L) + \frac{i}{2}(\gamma^\mu)_{bc}(D_a V_\mu) - \frac{1}{2}\left(\gamma^5\gamma^\mu\right)_{bc}(D_a U_\mu)$$
$$= X_{abc} + Y_{abc} + Z_{abc} + W_{abc} \tag{45}$$

$$X_{abc} = -i(\gamma^\mu)_{bc}\partial_\mu\{\rho_a - \zeta_a\} = -i(\gamma^\mu)_{bc}(\partial_\mu\rho_a) + i(\gamma^\mu)_{bc}(\partial_\mu\zeta_a) \tag{46}$$

$$Y_{abc} = (\gamma^5\gamma^\mu)_{bc}\partial_\mu\{-i(\gamma^5)_a{}^d(\rho_d + \zeta_d)\} = -i(\gamma^5\gamma^\mu)_{bc}(\gamma^5)_a{}^d(\partial_\mu\rho_d) - i(\gamma^5\gamma^\mu)_{bc}(\gamma^5)_a{}^d(\partial_\mu\zeta_d) \tag{47}$$

$$Z_{abc} = \frac{i}{2}(\gamma^\mu)_{bc}\left\{-(\gamma_\mu)_a{}^d\beta_d - (\gamma_\mu\gamma^\nu)_a{}^d(\partial_\nu\zeta_d) + (\partial_\mu\rho_a) + \frac{1}{2}(\gamma^\nu\gamma_\mu)_a{}^d(\partial_\nu\rho_d)\right\}$$
$$= -\frac{i}{2}(\gamma^\mu)_{bc}(\gamma_\mu)_a^d\beta_d - \frac{i}{2}(\gamma^\mu)_{bc}(\gamma_\mu\gamma^\nu)_a{}^d(\partial_\nu\zeta_d) + \frac{i}{2}(\gamma^\mu)_{bc}(\partial_\mu\rho_a) + \frac{i}{4}(\gamma^\mu)_{bc}(\gamma^\nu\gamma_\mu)_a{}^d(\partial_\nu\rho_d) \tag{48}$$

$$W_{abc} = -\frac{1}{2}\left(\gamma^5\gamma^\mu\right)_d \left\{ i\left(\gamma^5\gamma_\mu\right)_a{}^d\beta_d - i\left(\gamma^5\gamma_\mu\gamma^\nu\right)_a{}^d(\partial_\nu\zeta_d) - i\left(\gamma^5\right)_a{}^d(\partial_\mu\rho_d) - \frac{i}{2}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_a{}^d(\partial_\nu\rho_d)\right\}$$
$$= -\frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma_\mu\right)_a{}^d\beta_d + \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma_\mu\gamma^\nu\right)_a{}^d(\partial_\nu\zeta_d) + \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\right)_a{}^d(\partial_\mu\rho_d)$$
$$+\frac{i}{4}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_a{}^d(\partial_\nu\rho_d) \tag{49}$$

β terms with symmetry:

$$-\frac{i}{2}(\gamma^\mu)_{ec}(\gamma_\mu)_f{}^d\beta_d\delta_{(a}{}^e\delta_{b)}^f - \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{ec}(\gamma^5\gamma_\mu)_f{}^d\beta_d\delta_{(a}{}^e\delta_{b)}{}^f$$
$$= -\frac{i}{2}\beta_d\left\{-\frac{1}{2}(\gamma^\nu)_{ab}(\gamma_\nu)^{ef} - \frac{1}{4}\left(\sigma^{\nu\lambda}\right)_{ab}(\sigma_{\nu\lambda})^{ef}\right\}\left\{(\gamma^\mu)_{ec}(\gamma_\mu)_f{}^d + \left(\gamma^5\gamma^\mu\right)_{ec}\left(\gamma^5\gamma_\mu\right)_f{}^d\right\}$$
$$= \frac{i}{4}\beta_d(\gamma^\nu)_{ab}(\gamma^\mu)_{ce}(\gamma_\nu)^{ef}(\gamma_\mu)_f{}^d + \frac{i}{8}\beta_d(\sigma^{\nu\lambda})_{ab}(\gamma^\mu)_{ce}(\sigma_{\nu\lambda})^{ef}(\gamma_\mu)_f{}^d$$
$$-\frac{i}{4}\beta_d(\gamma^\nu)_{ab}(\gamma^\mu)_{ce}(\gamma_\nu)^{ef}(\gamma_\mu)_f{}^d - \frac{i}{8}\beta_d(\sigma^{\nu\lambda})_{ab}(\gamma^5\gamma^\mu)_{ce}(\sigma_{\nu\lambda})^{ef}(\gamma^5\gamma_\mu)_f{}^d$$
$$= -\frac{i}{4}\beta_d(\gamma^\nu)_{ab}(\gamma^\mu\gamma_\nu\gamma_\mu)_c{}^d - \frac{i}{8}\beta_d(\sigma^{\nu\lambda})_{ab}(\gamma^\mu\sigma_{\nu\lambda}\gamma_\mu)_c{}^d$$
$$+\frac{i}{4}\beta_d(\gamma^\nu)_{ab}(\gamma^5\gamma^\mu\gamma_\nu\gamma^5\gamma_\mu)_c{}^d + \frac{i}{8}\beta_d(\sigma^{\nu\lambda})_{ab}(\gamma^5\gamma^\mu\sigma_{\nu\lambda}\gamma^5\gamma_\mu)_c{}^d$$
$$= -\frac{i}{4}\beta_d\left(\sigma^{\nu\lambda}\right)_{ab}(\gamma^\mu\sigma_{\nu\lambda}\gamma_\mu)_c{}^d = 0 \tag{50}$$

ρ terms:

$$-\frac{i}{2}(\gamma^\mu)_{bc}(\partial_\mu\rho_a) - \frac{i}{2}(\gamma^5\gamma^\mu)_{bc}(\gamma^5)_a{}^d(\partial_\mu\rho_d) + \frac{i}{4}(\gamma^\mu)_{bc}(\gamma^\nu\gamma_\mu)_a{}^d(\partial_\nu\rho_d)$$
$$+\frac{i}{4}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_a{}^d(\partial_\nu\rho_d) \tag{51}$$

ρ terms with symmetry:

$$
\begin{aligned}
&= -\tfrac{i}{2}(\gamma^\mu)_{ec}(\partial_\mu\rho_f)\delta_{(a}{}^e\delta_{b)}{}^f - \tfrac{i}{2}(\gamma^5\gamma^\mu)_{ec}(\gamma^5)_f{}^d(\partial_\mu\rho_d)\delta_{(a}{}^e\delta_{b)}{}^f + \tfrac{i}{4}(\gamma^\mu)_{ec}(\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)\delta_{(a}{}^e\delta_{b)}{}^f \\
&+\tfrac{i}{4}(\gamma^5\gamma^\mu)_{ec}(\gamma^5\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)\delta_{(a}{}^e\delta_{b)}{}^f \\
&= \{-\tfrac{1}{2}(\gamma^\lambda)_{ab}(\gamma^\mu)_{ce}(\gamma_\lambda)^{ef} - \tfrac{1}{4}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}\} + \{\tfrac{1}{2}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu)_{ce}(\gamma_\lambda)^{ef} \\
&+\tfrac{1}{4}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}\}\{-\tfrac{i}{2}(\gamma^5)_f{}^d(\partial_\mu\rho_d) + \tfrac{i}{4}(\gamma^5\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)\} \\
&= \{\tfrac{1}{2}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda)_c{}^f + \tfrac{1}{4}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi})_c{}^f\}\{-\tfrac{i}{2}(\partial_\mu\rho_f) + \tfrac{i}{4}(\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)\} \\
&+\{-\tfrac{1}{2}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu\gamma_\lambda)_c{}^f - \tfrac{1}{4}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu\sigma_{\lambda\varphi})_c{}^f\}\{-\tfrac{i}{2}(\gamma^5)_f{}^d(\partial_\mu\rho_d) + \tfrac{i}{4}(\gamma^5\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)\} \\
&= -\tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda)_c{}^f(\partial_\mu\rho_f) - \tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi})_c{}^f(\partial_\mu\rho_f) + \tfrac{i}{8}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d) \\
&+\tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi}\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d)x + \tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu\gamma_\lambda\gamma^5)_c{}^d(\partial_\mu\rho_d) + \tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu\sigma_{\lambda\varphi}\gamma^5)_c{}^d(\partial_\mu\rho_d) \\
&-\tfrac{i}{8}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu\gamma_\lambda\gamma^5\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d) - \tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu\sigma_{\lambda\varphi}\gamma^5\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d) \\
&= -\tfrac{i}{4}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi})_c{}^d(\partial_\mu\rho_d) + \tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi}\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d) \\
&= -\tfrac{i}{4}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi})_c{}^d(\partial_\mu\rho_d) + \tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(2\delta_\mu{}^\nu\gamma^\mu\sigma_{\lambda\varphi} - \gamma^\mu\sigma_{\lambda\varphi}\gamma_\mu\gamma^\nu)_c{}^d(\partial_\nu\rho_d) = 0
\end{aligned}
\tag{52}
$$

ζ terms:

$$
\begin{aligned}
&i\,(\gamma^\mu)_{bc}\,(\partial_\mu\zeta_a) - i\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\right)_a{}^d\,(\partial_\mu\zeta_d) - \tfrac{i}{2}\,(\gamma^\mu)_{bc}\,(\gamma_\mu\gamma^\nu)_a{}^d\,(\partial_\nu\zeta_d) \\
&+\tfrac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma_\mu\gamma^\nu\right)_a{}^d\,(\partial_\nu\zeta_d)
\end{aligned}
\tag{53}
$$

ζ terms with symmetry:

$$i\left(\gamma^{\mu}\right)_{ec}\left(\partial_{\mu}\zeta_{p}\right)\delta_{(a}{}^{e}\delta_{b)}{}^{f}-i\left(\gamma^{5}\gamma^{\mu}\right)_{ec}\left(\gamma^{5}\right)_{f}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)\delta^{e}_{(a}\delta^{f}_{b)}-\frac{i}{2}\left(\gamma^{\mu}\right)_{ec}\left(\gamma_{\mu}\gamma^{\nu}\right)_{f}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)\delta^{e}_{(a}\delta^{f}_{b)}$$
$$+\frac{i}{2}\left(\gamma^{5}\gamma^{\mu}\right)_{ec}\left(\gamma^{5}\gamma_{\mu}\gamma^{\nu}\right)_{f}{}^{d}\left(\partial_{\nu}\zeta d\right)\left(\delta^{e}_{(a}\delta^{f}_{b)}\right)$$
$$=-\frac{i}{2}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\right)_{ce}\left(\gamma_{\lambda}\right)^{ef}\left(\partial_{\mu}\zeta_{f}\right)-\frac{i}{4}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{\mu}\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\partial_{\mu}\zeta_{f}\right)$$
$$-\frac{i}{2}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\right)_{ce}\left(\gamma_{\lambda}\right)^{ef}\left(\gamma^{5}\right)_{f}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)-\frac{i}{4}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma^{5}\right)_{f}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)$$
$$+\frac{i}{4}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\right)_{ce}\left(\gamma_{\lambda}\right)^{ef}\left(\gamma_{\mu}\gamma^{\nu}\right)_{f}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)+\frac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{\mu}\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma_{\mu}\gamma^{\nu}\right)_{f}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)$$
$$+\frac{i}{4}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\right)_{ce}\left(\gamma_{\lambda}\right)^{ef}\left(\gamma^{5}\gamma_{\mu}\gamma^{\nu}\right)_{f}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)+\frac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma^{5}\gamma_{\mu}\gamma^{\nu}\right)_{f}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)$$
$$=\frac{i}{2}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\gamma_{\lambda}\right)_{c}{}^{f}\left(\partial_{\mu}\zeta_{f}\right)+\frac{i}{4}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{\mu}\sigma_{\lambda\varphi}\right)_{c}{}^{f}\left(\partial_{\mu}\zeta_{f}\right)$$
$$+\frac{i}{2}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\gamma_{\lambda}\gamma^{5}\right)_{c}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)+\frac{i}{4}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\sigma_{\lambda\varphi}\gamma^{5}\right)_{c}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)$$
$$-\frac{i}{4}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\gamma_{\lambda}\gamma_{\mu}\gamma^{\nu}\right)_{c}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)-\frac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{\mu}\sigma_{\lambda\varphi}\gamma_{\mu}\gamma^{\nu}\right)_{c}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)$$
$$-\frac{i}{4}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\gamma_{\lambda}\gamma^{5}\gamma_{\mu}\gamma^{\nu}\right)_{c}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)-\frac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^{5}\gamma^{\mu}\sigma_{\lambda\varphi}\gamma^{5}\gamma_{\mu}\gamma^{\nu}\right)_{c}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)$$
$$=i\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\gamma_{\lambda}\right)_{c}{}^{d}\left(\partial_{\mu}\zeta_{\lambda}\right)-\frac{i}{2}\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\gamma_{\lambda}\gamma_{\mu}\gamma^{\nu}\right)_{c}{}^{d}\left(\partial_{\nu}\zeta_{d}\right)$$
$$=i\left(\gamma^{\lambda}\right)_{ab}\left(\gamma^{\mu}\gamma_{\lambda}\right)_{c}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)+i(\gamma^{\lambda})_{ab}\left(\gamma_{\lambda}\gamma^{\mu}\right)_{c}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)$$
$$=2i\left(\gamma^{\lambda}\right)_{ab}\delta_{\lambda}{}^{\mu}\delta_{c}{}^{d}\left(\partial_{\mu}\zeta_{d}\right)=2i\left(\gamma^{\lambda}\right)_{ab}\left(\partial_{\lambda}\zeta_{c}\right) \tag{54}$$

$$\{D_a,D_b\}\zeta_c=2i\left(\gamma^{\lambda}\right)_{ab}\left(\partial_{\lambda}\zeta_c\right) \tag{55}$$

1.8 ρ_c field

$$D_aD_b\rho_c=iC_{bc}\left(D_aM\right)+\left(\gamma^{5}\right)_{bc}\left(D_aN\right)+\frac{i}{2}\left(\gamma^{\mu}\right)_{bc}\left(D_aV_{\mu}\right)+\frac{1}{2}\left(\gamma^{5}\gamma^{\mu}\right)_{bc}\left(D_aU_{\mu}\right)$$
$$=X_{abc}+Y_{abc}+Z_{abc}+W_{abc} \tag{56}$$

$$X_{abc}=iC_{bc}\left(\beta_a-\frac{1}{2}\left(\gamma^{\mu}\right)_{a}{}^{d}\left(\partial_{\mu}\rho_d\right)\right)=iC_{bc}\beta_a-\frac{i}{2}C_{bc}\left(\gamma^{\mu}\right)_{a}{}^{d}\left(\partial_{\mu}\rho_d\right) \tag{57}$$

$$Y_{abc}=-i\left(\gamma^{5}\right)_{bc}\left(\gamma^{5}\right)_{a}{}^{d}\beta_d+\frac{i}{2}\left(\gamma^{5}\right)_{bc}\left(\gamma^{5}\gamma^{\mu}\right)_{a}{}^{d}\left(\partial_{\mu}\rho_d\right) \tag{58}$$

$$Z_{abc}=-\frac{i}{2}\left(\gamma^{\mu}\right)_{bc}\left(\gamma_{\mu}\right)_{a}{}^{d}\beta_d-\frac{i}{2}\left(\gamma^{\mu}\right)_{bc}\left(\gamma_{\mu}\gamma^{\nu}\right)_{a}{}^{d}\left(\partial_{\nu}\zeta_d\right)+\frac{i}{2}\left(\gamma^{\mu}\right)_{bc}\left(\partial_{\mu}\rho_a\right)+\frac{i}{4}\left(\gamma^{\mu}\right)_{bc}\left(\gamma^{\nu}\gamma_{\mu}\right)_{a}{}^{d}(\partial_{\nu}\rho_d) \tag{59}$$

$$W_{abc} = \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma_\mu\right)_a{}^d\beta_d - \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma_\mu\gamma^\nu\right)_a{}^d\left(\partial_\nu\zeta_d\right) - \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\right)_a{}^d\left(\partial_\mu\rho_d\right)$$
$$-\frac{i}{4}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\gamma^5\gamma^\nu\gamma_\mu\right)_a{}^d\left(\partial_\nu\rho_d\right) \tag{60}$$

ζ terms with symmetry:

$$-\frac{i}{2}\left(\gamma^\mu\right)_{ec}\left(\gamma_\mu\gamma^\nu\right)_f^d\left(\partial_\nu\zeta_d\right)\delta_{(a}{}^e\delta_{b)}{}^f - \frac{i}{2}\left(\gamma^5\gamma^\mu\right)_{ec}\left(\gamma^5\gamma_\mu\gamma^\nu\right)_f{}^d\left(\partial_\nu\zeta_d\right)\delta^e_{(a}\delta^f_{b)}$$
$$= \frac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma_\mu\gamma^\nu\right)_f{}^d\left(\partial_\nu\zeta_d\right) + \frac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^\mu\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma_\mu\gamma^\nu\right)_f{}^d\left(\partial_\nu\zeta_d\right)$$
$$-\frac{i}{4}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu)_{ce}(\gamma_\lambda)^{ef}(\gamma^5\gamma_\mu\gamma^\nu)_f{}^d(\partial_\nu\zeta_d) - \frac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma^5\gamma_\mu\gamma^\nu)_f{}^d(\partial_\nu\zeta_d)$$
$$= -\frac{i}{4}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda\gamma_\mu\gamma^\nu)_c{}^d(\partial_\nu\zeta_d) - \frac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi}\gamma_\mu\gamma^\nu)_c{}^d(\partial_\nu\zeta_d)$$
$$+\frac{i}{4}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu\gamma_\lambda\gamma^5\gamma_\mu\gamma^\nu)_c{}^d(\partial_\nu\zeta_d) - \frac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu\sigma_{\lambda\varphi}\gamma^5\gamma_\mu\gamma^\nu)_c{}^d(\partial_\nu\zeta_d)$$
$$= -\frac{i}{4}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^\mu\sigma_{\lambda\varphi}\gamma_\mu\gamma^\nu\right)_c{}^d\left(\partial_\nu\zeta_d\right) = 0 \tag{61}$$

β terms with symmetry:

$$\{iC_{ec}\beta_f - i(\gamma^5)_{ec}(\gamma^5)_f{}^d\beta_d - \frac{i}{2}(\gamma^\mu)_{ec}(\gamma_\mu)_f{}^d\beta_d + \frac{i}{2}(\gamma^5\gamma^\mu)_{ec}(\gamma^5\gamma_\mu)_f{}^d\beta_d\}\delta_{(a}{}^e\delta_{b)}{}^f$$
$$= -\frac{i}{2}\left(\gamma^\nu\right)_{ab}C_{ec}\left(\gamma_\nu\right)^{ef}\beta_f - \frac{i}{4}\left(\sigma^{\nu\lambda}\right)_{ab}C_{ec}\left(\sigma_{\nu\lambda}\right)^{ef}\beta_f$$
$$-\frac{i}{2}\left(\gamma^\nu\right)_{ab}\left(\gamma^5\right)_{ce}\left(\gamma_\nu\right)^{ef}\left(\gamma^5\right)_f{}^d\beta_d - \frac{i}{4}\left(\sigma^{\nu\lambda}\right)_{ab}\left(\gamma^5\right)_{ce}\left(\sigma_{\nu\lambda}\right)^{ef}\left(\gamma^5\right)_f{}^d\beta_d$$
$$+\frac{i}{4}\left(\gamma^\nu\right)_{ab}\left(\gamma^\mu\right)_{ce}\left(\gamma_\nu\right)^{ef}\left(\gamma_\mu\right)_f{}^d\beta_d + \frac{i}{8}\left(\sigma^{\nu\lambda}\right)_{ab}\left(\gamma^\mu\right)_{ce}\left(\sigma_{\nu\lambda}\right)^{ef}\left(\gamma_\mu\right)_f{}^d\beta_d$$
$$+\frac{i}{4}\left(\gamma^\nu\right)_{ab}\left(\gamma^5\gamma^\mu\right)_{ce}\left(\gamma_\nu\right)^{ef}\left(\gamma^5\gamma_\mu\right)_f{}^d\beta_d + \frac{i}{8}\left(\sigma^{\nu\lambda}\right)_{ab}\left(\gamma^5\gamma^\mu\right)_{ce}\left(\sigma_{\nu\lambda}\right)^{ef}\left(\gamma^5\gamma_\mu\right)_f{}^d\beta_d$$
$$= -\frac{i}{2}\left(\gamma^\nu\right)_{ab}\left(\gamma_\nu\right)_c{}^f\beta_f - \frac{i}{4}\left(\sigma^{\nu\lambda}\right)_{ab}\left(\sigma_{\nu\lambda}\right)_c{}^f\beta_f$$
$$+\frac{i}{2}\left(\gamma^\nu\right)_{ab}\left(\gamma^5\gamma_\nu\gamma^5\right)_c{}^d\beta_d + \frac{i}{4}\left(\sigma^{\nu\lambda}\right)_{ab}\left(\gamma^5\sigma_{\nu\lambda}\gamma^5\right)_c{}^d\beta_d$$
$$-\frac{i}{4}\left(\gamma^\nu\right)_{ab}\left(\gamma^\mu\gamma_\nu\gamma_\mu\right)_c{}^d\beta_d - \frac{i}{8}\left(\sigma^{\nu\lambda}\right)_{ab}(\gamma^\mu\sigma_{\nu\lambda}\gamma_\mu)_c{}^d\beta_d$$
$$-\frac{i}{4}\left(\gamma^\nu\right)_{ab}\left(\gamma^5\gamma^\mu\gamma_\nu\gamma^5\gamma_\mu\right)_c{}^{pd}\beta_d - \frac{i}{8}\left(\sigma^{\nu\lambda}\right)_{ab}\left(\gamma^5\gamma^\mu\sigma_{\nu\lambda}\gamma^5\gamma_\mu\right)_c{}^d\beta_d$$
$$= -i\left(\gamma^\nu\right)_{ab}\left(\gamma_\nu\right)_c{}^d\beta_d - \frac{i}{2}\left(\gamma^\nu\right)_{ab}\left(\gamma^\mu\gamma_\nu\gamma_\mu\right)_c{}^d\beta_d$$
$$= -i\left(\gamma^\nu\right)_{ab}\left(\gamma_\nu\right)_c{}^d\beta_d + i\left(\gamma^\nu\right)_{ab}\left(\gamma_\nu\right)_c{}^d\beta_d = 0 \tag{62}$$

ρ terms with symmetry:

$$\{-\tfrac{i}{2}C_{ec}(\gamma^\mu)_f{}^d(\partial_\mu\rho_d)+\tfrac{i}{2}(\gamma^5\gamma^\mu)_f{}^d(\partial_\mu\rho_d)+\tfrac{i}{2}(\gamma^\mu)_{ec}(\partial_\mu\rho_f)+\tfrac{i}{4}(\gamma^\mu)_{ec}(\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)$$
$$-\tfrac{i}{2}(\gamma^5\gamma^\mu)_{ec}(\gamma^5)_f{}^d(\partial_\mu\rho_d)-\tfrac{i}{4}(\gamma^5\gamma^\mu)_{ec}(\gamma^5\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)\}\delta_{(a}{}^e\delta_{b)}{}^f$$
$$=\tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}C_{ec}\left(\gamma_\lambda\right)^{ef}\left(\gamma^\mu\right)_f{}^d\left(\partial_\mu\rho_d\right)+\tfrac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}C_{ec}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma^\mu\right)_f{}^d\left(\partial_\mu\rho_d\right)$$
$$+\tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^5)_{ce}(\gamma_\lambda)^{ef}(\gamma^5\gamma^\mu)_f{}^d(\partial_\mu\rho_d)+\tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma^5\gamma^\mu)_f{}^d(\partial_\mu\rho_d)$$
$$-\tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\partial_\mu\rho_f\right)-\tfrac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^\mu\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\partial_{\mu\rho_f}\right)$$
$$-\tfrac{i}{8}(\gamma^\lambda)_{ab}(\gamma^\mu)_{ce}(\gamma_\lambda)^{ef}(\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)-\tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)$$
$$-\tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^5\gamma^\mu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\rho_d\right)-\tfrac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^5\gamma^\mu\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\rho_d\right)$$
$$-\tfrac{i}{8}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu)_{ce}(\gamma_\lambda)^{ef}(\gamma^5\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)-\tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma^5\gamma^\nu\gamma_\mu)_f{}^d(\partial_\nu\rho_d)$$
$$=\tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma_\lambda\gamma^\mu\right)_c{}^d\left(\partial_\mu\rho_d\right)+\tfrac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\sigma_{\lambda\varphi}\gamma^\mu\right)_c{}^d(\partial_\mu\rho_d)$$
$$-\tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^5\gamma_\lambda\gamma^5\gamma^\mu)_c{}^d(\partial_\mu\rho_d)-\tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\sigma_{\lambda\varphi}\gamma^5\gamma^\mu)_c{}^d(\partial_\mu\rho_d)$$
$$+\tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\gamma_\lambda\right)_c{}^d\left(\partial_\mu\rho_d\right)+\tfrac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^\mu\sigma_{\lambda\varphi}\right)_c{}^d\left(\partial_\mu\rho_d\right)$$
$$+\tfrac{i}{8}\left(\gamma^\lambda\right)_{ab}(\gamma^\mu\gamma_\lambda\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d)+\tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\sigma_{\lambda\varphi}\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d)$$
$$+\tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^5\gamma^\mu\gamma_\lambda\gamma^5)_c{}^d(\partial_\mu\rho_d)+\tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu\sigma_{\lambda\varphi}\gamma^5)_c{}^d(\partial_\mu\rho_d)$$
$$+\tfrac{i}{8}\left(\gamma^\lambda\right)_{ab}\left(\gamma^5\gamma^\mu\gamma_\lambda\gamma^5\gamma^\nu\gamma_\mu\right)_c{}^d\left(\partial_\nu\rho_d\right)+\tfrac{i}{16}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^5\gamma^\mu\sigma_{\lambda\varphi}\gamma^5\gamma^\nu\gamma_\mu\right)_c{}^d\left(\partial_\nu\rho_d\right)$$
$$=\tfrac{i}{2}(\gamma^\lambda)_{ab}(\gamma_\lambda\gamma^\mu)_c{}^d(\partial_\mu\rho_d)+\tfrac{i}{2}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda)_c{}^d(\partial_\mu\rho_d)+\tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda\gamma^\nu\gamma_\mu)_c{}^d(\partial_\nu\rho_d)$$
$$=i\delta_\lambda{}^\mu\delta_c{}^d(\gamma^\lambda)_{ab}(\partial_\mu\rho_d)+\tfrac{i}{4}\eta^{\nu\varphi}(\gamma^\lambda)_{ab}(4\eta_{\lambda\varphi}\mathbf{I})_c{}^d(\partial_\nu\rho_d)$$
$$=i\left(\gamma^\lambda\right)_{ab}\left(\partial_\lambda\rho_c\right)+i(\gamma^\lambda)_{ab}(\partial_\lambda\rho_c)=2i\left(\gamma^\lambda\right)_{ab}\left(\partial_\lambda\rho_c\right) \tag{63}$$

$$\{D_a,D_b\}\rho_c=2i\left(\gamma^\lambda\right)_{ab}\left(\partial_\lambda\rho_c\right) \tag{64}$$

1.9 β_c field

$$D_aD_b\beta_c=-\eta^{\mu\nu}\partial_\mu\partial_\nu\left\{iC_{bc}\left(D_aK\right)+\left(\gamma^5\right)_{bc}\left(D_aL\right)\right\}+\tfrac{i}{2}\left(\gamma^\mu\right)_{bc}\left(\partial_\mu D_aM\right)+\tfrac{1}{2}\left(\gamma^5\gamma^\mu\right)_{bc}\left(\partial_\mu D_aN\right)$$
$$+\tfrac{i}{2}\left(\gamma^\mu\gamma^\nu\right)_{bc}\left(\partial_\mu D_aV_\nu\right)+\tfrac{i}{4}\left(\gamma^\nu\gamma^\mu\right)_{bc}\left(\partial_\mu D_aV_\nu\right)+\tfrac{1}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{bc}\left(\partial_\mu D_aU_\nu\right)+\tfrac{1}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{bc}\left(\partial_\mu D_aU_\nu\right)$$
$$A_{abc}+B_{abc}+C_{abc}+D_{abc}+E_{abc}+F_{abc}+G_{abc}+H_{abc} \tag{65}$$

$$A_{abc} = -i\eta^{\mu\nu} C_{bc} \left(\partial_\mu \partial_\nu \rho_a\right) + i\eta^{\mu\nu} C_{bc} \left(\partial_\mu \partial_\nu \zeta_a\right)$$

$$B_{abc} = i\eta^{\mu\nu} \left(\gamma^5\right)_{bc} \left(\gamma^5\right)_a{}^d \left(\partial_\mu \partial_\nu \rho_d\right) + i\eta^{\mu\nu} \left(\gamma^5\right)_{bc} \left(\gamma^5\right)_a{}^d \left(\partial_\mu \partial_\nu \zeta_d\right)$$

$$C_{abc} = \tfrac{i}{2} \left(\gamma^\mu\right)_{bc} \left(\partial_\mu \beta_a\right) - \tfrac{i}{4} \left(\gamma^\mu\right)_{bc} \left(\gamma^\nu\right)_a{}^d \left(\partial_\mu \partial_\nu \rho_d\right)$$

$$D_{abc} = -\tfrac{i}{2} \left(\gamma^5\gamma^\mu\right)_{bc} \left(\gamma^5\right)_a{}^d \left(\partial_\mu \beta_d\right) + \tfrac{i}{4} \left(\gamma^5\gamma^\mu\right)_{bc} \left(\gamma^5\gamma^\nu\right)_a{}^d \left(\partial_\mu \partial_\nu \rho_d\right)$$

$$E_{abc} = -\tfrac{i}{2} \left(\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma_\nu\right)_a{}^d \left(\partial_\mu \beta_d\right) - \tfrac{i}{2} \left(\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma_\nu\gamma^\lambda\right)_a{}^d \left(\partial_\mu \partial_\lambda \zeta_d\right) + \tfrac{i}{2} \left(\gamma^\mu\gamma^\nu\right)_{bc} \left(\partial_\mu \partial_\nu \rho_a\right)$$
$$+\tfrac{i}{4} \left(\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma^\lambda\gamma_\nu\right)_a{}^d \left(\partial_\mu \partial_\lambda \rho_d\right)$$

$$F_{abc} = -\tfrac{i}{4} \left(\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma_\nu\right)_a{}^d \left(\partial_\mu \beta_d\right) - \tfrac{i}{4} \left(\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma_\nu\gamma^\lambda\right)_a{}^d \left(\partial_\mu \partial_\lambda \zeta_d\right) + \tfrac{i}{4} \left(\gamma^\nu\gamma^\mu\right)_{bc} \left(\partial_\mu \partial_\nu \rho_a\right)$$
$$+\tfrac{i}{8} \left(\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma^\lambda\gamma_\nu\right)_a{}^d \left(\partial_\mu \partial_\lambda \rho_d\right)$$

$$G_{abc} = \tfrac{i}{2} \left(\gamma^5\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma^5\gamma_\nu\right)_a{}^d \left(\partial_\mu \beta_d\right) - \tfrac{i}{2} \left(\gamma^5\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma^5\gamma_\nu\gamma^\lambda\right)_a{}^d \left(\partial_\mu \partial_\lambda \zeta_d\right)$$
$$-\tfrac{i}{2} \left(\gamma^5\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma^5\right)_a{}^d \left(\partial_\mu \partial_\nu \rho_d\right) - \tfrac{i}{4} \left(\gamma^5\gamma^\mu\gamma^\nu\right)_{bc} \left(\gamma^5\gamma^\lambda\gamma_\nu\right)_a{}^d \left(\partial_\mu \partial_\lambda \rho_d\right)$$

$$H_{abc} = \tfrac{i}{4} \left(\gamma^5\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma^5\gamma_\nu\right)_a{}^d \left(\partial_\mu \beta_d\right) - \tfrac{i}{4} \left(\gamma^5\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma^5\gamma_\nu\gamma^\lambda\right)_a{}^d \left(\partial_\mu \partial_\lambda \zeta_d\right)$$
$$-\tfrac{i}{4} \left(\gamma^5\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma^5\right)_a{}^d \left(\partial_\mu \partial_\nu \rho_d\right) - \tfrac{i}{8} \left(\gamma^5\gamma^\nu\gamma^\mu\right)_{bc} \left(\gamma^5\gamma^\lambda\gamma_\nu\right)_a{}^d \left(\partial_\mu \partial_\lambda \rho_d\right) \tag{66}$$

ζ terms with symmetry:

$$\{i\eta^{\mu\nu} C_{ec} \left(\partial_\mu \partial_\nu \zeta_f\right) + i\eta^{\mu\nu} \left(\gamma^5\right)_{ec} \left(\gamma^5\right)_f{}^d \left(\partial_\mu \partial_\nu \zeta_d\right) - \tfrac{i}{2} \left(\gamma^\mu\gamma^\nu\right)_{ec} \left(\gamma_\nu\gamma^\lambda\right)_f{}^d \left(\partial_\mu \partial_\lambda \zeta_d\right)$$
$$-\tfrac{i}{4}(\gamma^\nu\gamma^\mu)_{ec}(\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d) - \tfrac{i}{2}(\gamma^5\gamma^\mu\gamma^\nu)_{ec}(\gamma^5\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d)$$
$$-\tfrac{i}{4} \left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ec} \left(\gamma^5\gamma_\nu\gamma^\lambda\right)_f{}^d \left(\partial_\mu \partial_\lambda \zeta_d\right)\}\delta_{(a}{}^e\delta_{b)}{}^f \tag{67}$$

$$= -\tfrac{i}{2}\eta^{\mu\nu}\left(\gamma^{\varphi}\right)_{ab}C_{ec}\left(\gamma_{\varphi}\right)^{ef}\left(\partial_\mu\partial_\nu\zeta_f\right) - \tfrac{i}{4}\eta^{\mu\nu}\left(\sigma^{\varphi\theta}\right)_{ab}C_{ec}\left(\sigma_{\varphi\theta}\right)^{ef}\left(\partial_\mu\partial_\nu\zeta_f\right)$$
$$+\tfrac{i}{2}\eta^{\mu\nu}(\gamma^{\varphi})_{ab}(\gamma^5)_{ce}(\gamma_{\varphi})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\zeta_d) + \tfrac{i}{4}\eta^{\mu\nu}(\sigma^{\varphi\theta})_{ab}(\gamma^5)_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\zeta_d)$$
$$-\tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^\nu\gamma^\mu)_{ce}(\gamma_{\varphi})^{ef}(\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d) - \tfrac{i}{8}(\sigma^{\varphi\theta})_{ab}(\gamma^\nu\gamma^\mu)_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d)$$
$$-\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^\mu\gamma^\nu)_{ce}(\gamma_{\varphi})^{ef}(\gamma_\nu\gamma^\lambda)_f{}^d((\partial_\mu\partial_\lambda\zeta_d) - \tfrac{i}{16}(\sigma^{\varphi\theta})_{ab}(\gamma^\mu\gamma^\nu)_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d)$$
$$-\tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^\nu\gamma^\mu)_{ce}(\gamma_{\varphi})^{ef}(\gamma^5\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d) - \tfrac{i}{8}(\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^\nu\gamma^\mu)_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d)$$
$$-\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^\mu\gamma^\nu)_{ce}(\gamma_{\varphi})^{ef}(\gamma^5\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d) - \tfrac{i}{16}(\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^\mu\gamma^\nu)_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5\gamma_\nu\gamma^\lambda)_f{}^d(\partial_\mu\partial_\lambda\zeta_d)$$

(68)

$$= -\tfrac{i}{2}\eta^{\mu\nu}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\right)_c{}^d\left(\partial_\mu\partial_\nu\zeta_d\right) + \tfrac{i}{4}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\gamma^\nu\gamma^\mu\sigma_{\varphi\theta}\gamma_\nu\gamma^\lambda\right)_c{}^d\left(\partial_\mu\partial_\lambda\zeta_d\right)$$
$$+\tfrac{i}{8}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\gamma^\mu\gamma^\nu\sigma_{\varphi\theta}\gamma_\nu\gamma^\lambda\right)_c{}^d\left(\partial_\mu\partial_\lambda\zeta_d\right)$$
$$= -\tfrac{i}{2}\eta^{\mu\nu}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\right)_c{}^d\left(\partial_\mu\partial_\nu\zeta_d\right) + \tfrac{i}{2}\eta^{\nu\mu}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\gamma_\nu\gamma^\lambda\right)_c{}^d\left(\partial_\mu\partial_\lambda\zeta_d\right)$$
$$-\tfrac{i}{8}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\gamma^\mu\gamma^\nu\sigma_{\varphi\theta}\gamma_\nu\gamma^\lambda\right)_c{}^d\left(\partial_\mu\partial_\lambda\zeta_d\right)$$
$$= -\tfrac{i}{2}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\right)_c{}^d\left(\Box\zeta_d\right) + \tfrac{i}{2}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\gamma^\mu\gamma^\lambda\right)_c{}^d\left(\partial_\mu\partial_\lambda\zeta_d\right)$$
$$= -\tfrac{i}{2}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\right)_c{}^d\left(\Box\zeta_d\right) + \tfrac{i}{2}\eta^{\mu\lambda}\left(\sigma^{\varphi\theta}\right)_{ab}\left(\sigma_{\varphi\theta}\right)_c{}^d\left(\partial_\mu\partial_\lambda\zeta_d\right) = 0$$

(69)

ρ terms with symmetry:

$$\{-i\eta^{\mu\nu}C_{ec}\left(\partial_\mu\partial_\nu\rho_f\right) + i\eta^{\mu\nu}\left(\gamma^5\right)_{ec}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\partial_\nu\rho_d\right) - \tfrac{i}{4}\left(\gamma^\mu\right)_{ec}\left(\gamma^\nu\right)_f{}^d\left(\partial_\mu\partial_\nu\rho_d\right)$$
$$+\tfrac{i}{4}\left(\gamma^5\gamma^\mu\right)_{ec}\left(\gamma^5\gamma^\nu\right)_f{}^d\left(\partial_\mu\partial_\nu\rho_d\right) + \tfrac{i}{2}\left(\gamma^\mu\gamma^\nu\right)_{ec}\left(\partial_\mu\partial_\nu\rho_f\right) + \tfrac{i}{4}\left(\gamma^\mu\gamma^\nu\right)_{ec}\left(\gamma^\lambda\gamma_\nu\right)_f{}^d\left(\partial_\mu\partial_\lambda\rho_d\right)$$
$$+\tfrac{i}{4}(\gamma^\nu\gamma^\mu)_{ec}(\partial_\mu\partial_\nu\rho_f) + \tfrac{i}{8}(\gamma^\nu\gamma^\mu)_{ec}(\gamma^\lambda\gamma_\nu)_f{}^d(\partial_\mu\partial_\lambda\rho_d) - \tfrac{i}{2}(\gamma^5\gamma^\mu\gamma^\nu)_{ec}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d)$$
$$-\tfrac{i}{4}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ec}\left(\gamma^5\gamma^\lambda\gamma_\nu\right)_f{}^d\left(\partial_\mu\partial_\lambda\rho_d\right) - \tfrac{i}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ec}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\partial_\nu\rho_d\right)$$
$$-\tfrac{i}{8}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ec}\left(\gamma^5\gamma^\lambda\gamma_\nu\right)_f{}^d\left(\partial_\mu\partial_\lambda\rho_d\right)\}\delta_{(a}{}^e\delta_{b)}{}^f$$

(70)

$$
\begin{aligned}
&= \tfrac{i}{2}\eta^{\mu\nu}(\gamma^{\varphi})_{ab}C_{ec}(\gamma_{\varphi})^{ef}(\partial_\mu\partial_\nu\rho_f) + \tfrac{i}{4}\eta^{\mu\nu}(\sigma^{\varphi\theta})_{ab}C_{ec}(\sigma_{\varphi\theta})^{ef}(\partial_\mu\partial_\nu\rho_f) \\
&+\tfrac{i}{2}\eta^{\mu\nu}(\gamma^{\varphi})_{ab}(\gamma^5)_{ce}(\gamma_{\varphi})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d) + \tfrac{i}{4}\eta^{\mu\nu}(\sigma^{\varphi\theta})_{ab}(\gamma^5)_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d) \\
&+\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^{\mu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^{\nu})_f{}^d(\partial_\mu\partial_\nu\rho_d) + \tfrac{i}{16}(\sigma^{\varphi\theta})_{ab}(\gamma^{\mu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^{\nu})_f{}^d(\partial_\mu\partial_\nu\rho_d) \\
&+\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^{\mu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^5\gamma^{\nu})_f{}^d(\partial_\mu\partial_\nu\rho_d) + \tfrac{i}{16}\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^{\mu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5\gamma^{\nu})_f{}^d(\partial_\mu\partial_\nu\rho_d) \\
&+\tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^{\nu}\gamma^{\mu})_{ce}(\gamma_{\varphi})^{ef}(\partial_\mu\partial_\nu\rho_f) + \tfrac{i}{8}(\sigma^{\varphi\theta})_{ab}(\gamma^{\nu}\gamma^{\mu})_{ce}(\sigma_{\varphi\theta})^{ef}(\partial_\mu\partial_\nu\rho_f) \\
&+\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^{\nu}\gamma^{\mu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) + \tfrac{i}{16}\sigma^{\varphi\theta})_{ab}(\gamma^{\nu}\gamma^{\mu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) \\
&+\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^{\mu}\gamma^{\nu})_{ce}(\gamma_{\varphi})^{ef}(\partial_\mu\partial_\nu\rho_f) + \tfrac{i}{16}\sigma^{\varphi\theta})_{ab}(\gamma^{\mu}\gamma^{\nu})_{ce}(\sigma_{\varphi\theta})^{ef}(\partial_\mu\partial_\nu\rho_f) \\
&+\tfrac{i}{16}(\gamma^{\varphi})_{ab}(\gamma^{\mu}\gamma^{\nu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) + \tfrac{i}{32}(\sigma^{\varphi\theta})_{ab}(\gamma^{\mu}\gamma^{\nu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) \\
&-\tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^{\nu}\gamma^{\mu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d) - \tfrac{i}{8}(\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^{\nu}\gamma^{\mu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d) \\
&-\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^{\nu}\gamma^{\mu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^5\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) - \tfrac{i}{16}(\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^{\nu}\gamma^{\mu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) \\
&-\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^{\mu}\gamma^{\nu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d) - \tfrac{i}{16}(\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^{\mu}\gamma^{\nu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5)_f{}^d(\partial_\mu\partial_\nu\rho_d) \\
&-\tfrac{i}{16}(\gamma^{\varphi})_{ab}(\gamma^5\gamma^{\mu}\gamma^{\nu})_{ce}(\gamma_{\varphi})^{ef}(\gamma^5\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) - \tfrac{i}{32}(\sigma^{\varphi\theta})_{ab}(\gamma^5\gamma^{\mu}\gamma^{\nu})_{ce}(\sigma_{\varphi\theta})^{ef}(\gamma^5\gamma^{\lambda}\gamma_{\nu})_f{}^d(\partial_\mu\partial_\lambda\rho_d) \\
&= i\eta^{\mu\nu}(\gamma^{\varphi})_{ab}(\gamma_{\varphi})_c{}^d(\partial_\mu\partial_\nu\rho_d) - \tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^{\mu}\gamma_{\varphi}\gamma^{\nu})_c{}^d(\partial_\mu\partial_\nu\rho_d) - \tfrac{i}{2}(\gamma^{\varphi})_{ab}(\gamma^{\nu}\gamma^{\mu}\gamma_{\varphi})_c{}^d(\partial_\mu\partial_\nu\rho_d) \\
&-\tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^{\nu}\gamma^{\mu}\gamma_{\varphi}\gamma^{\lambda}\gamma_{\nu})_c{}^d(\partial_\mu\partial_\lambda\rho_d) - \tfrac{i}{4}(\gamma^{\varphi})_{ab}(\gamma^{\mu}\gamma^{\nu}\gamma_{\varphi})_c{}^d(\partial_\mu\partial_\nu\rho_d) \\
&-\tfrac{i}{8}(\gamma^{\varphi})_{ab}(\gamma^{\mu}\gamma^{\nu}\gamma_{\varphi}\gamma^{\lambda}\gamma_{\nu})_c{}^d(\partial_\mu\partial_\lambda\rho_d) \\
&= i\eta^{\mu\nu}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma_{\varphi}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) - \tfrac{i}{2}\delta_{\varphi}{}^{\nu}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) - \tfrac{i}{2}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\nu}\gamma^{\mu}\gamma_{\varphi}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) \\
&-\tfrac{i}{2}\eta^{\mu\nu}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma_{\varphi}\gamma^{\lambda}\gamma_{\nu}\right)_c{}^d\left(\partial_\mu\partial_\lambda\rho_d\right) + \tfrac{i}{8}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\mu}\gamma^{\nu}\gamma_{\varphi}\gamma^{\lambda}\gamma_{\nu}\right)_c{}^d\left(\partial_\mu\partial_\lambda\rho_d\right) \\
&= i\left(\gamma^{\varphi}\right)_{ab}\left(\gamma_{\varphi}\right)_c{}^d(\Box\rho_d) - \tfrac{i}{2}\left(\gamma^{\nu}\right)_{ab}\left(\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) - \tfrac{i}{2}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\nu}\gamma^{\mu}\gamma_{\varphi}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) \\
&-\tfrac{i}{2}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma_{\varphi}\gamma^{\nu}\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) + \tfrac{i}{8}\eta^{\lambda\theta}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\mu}\left(4\eta_{\varphi\theta}\mathbf{I}\right)\right)_c{}^d(\partial_\mu\partial_\lambda\rho_d) \\
&= i\left(\gamma^{\varphi}\right)_{ab}\left(\gamma_{\varphi}\right)_c{}^d\left(\Box\rho_d\right) - \tfrac{i}{2}\left(\gamma^{\nu}\right)_{ab}\left(\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) - \tfrac{i}{2}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\nu}\gamma^{\mu}\gamma_{\varphi}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) \\
&-i\delta_{\varphi}{}^{\nu}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) + \tfrac{i}{2}\left(\gamma^{\varphi}\right))_{ab}\left(\gamma^{\nu}\gamma_{\varphi}\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right) + \tfrac{i}{2}\delta_{\varphi}{}^{\lambda}\left(\gamma^{\varphi}\right)_{ab}\left(\gamma^{\mu}\right)_c{}^d\left(\partial_\mu\partial_\lambda\rho_d\right)
\end{aligned}
\tag{71}
$$

$$= i(\gamma^\varphi)_{ab}(\gamma_\varphi)_c{}^d(\Box\rho_d) - i(\gamma^\nu)_{ab}(\gamma^\mu)_c{}^d(\partial_\mu\partial_\nu\rho_d) - i(\gamma^\varphi)_{ab}(\gamma^\nu\gamma^\mu\gamma_\varphi)_c{}^d(\partial_\mu\partial_\nu\rho_d)$$
$$+ i\delta_\varphi{}^\mu(\gamma^\varphi)_{ab}(\gamma^\nu)_c{}^d(\partial_\mu\partial_\nu\rho_d)$$
$$= i\left(\gamma^\varphi\right)_{ab}\left(\gamma_\varphi\right)_c{}^d\left(\Box\rho_d\right) - i\left(\gamma^\varphi\right)_{ab}\left(\gamma^\nu\gamma^\mu\gamma_\varphi\right)_c{}^d\left(\partial_\mu\partial_\nu\rho_d\right)$$
$$= i\left(\gamma^\varphi\right)_{ab}\left(\gamma_\varphi\right)_c{}^d\left(\Box\rho_d\right) - i\eta^{\nu\mu}(\gamma^\varphi)_{ab}(\gamma_\varphi)_c{}^d(\partial_\mu\partial_\nu\rho_d) = 0 \tag{72}$$

β terms with symmetry:

$$\{\tfrac{i}{2}\left(\gamma^\mu\right)_{ec}\left(\partial_\mu\beta_f\right) - \tfrac{i}{2}\left(\gamma^5\gamma^\mu\right)_{ec}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\beta_d\right) - \tfrac{i}{2}\left(\gamma^\mu\gamma^\nu\right)_{ec}\left(\gamma_\nu\right)_f{}^d\left(\partial_\mu\beta_d\right)$$
$$- \tfrac{i}{4}\left(\gamma^\nu\gamma^\mu\right)_{ec}\left(\gamma_\nu\right)_f{}^d\left(\partial_\mu\beta_d\right) + \tfrac{i}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ec}\left(\gamma^5\gamma_\nu\right)_f{}^d\left(\partial_\mu\beta_d\right)$$
$$+ \tfrac{i}{4}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ec}\left(\gamma^5\gamma_\nu\right)_f{}^d\left(\partial_\mu\beta_d\right)\}\delta_{(a}{}^e\delta_{b)}{}^f$$
$$= -\tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^\mu)_{ce}(\gamma_\lambda)^{ef}(\partial_\mu\beta_f) - \tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\partial_\mu\beta_f)$$
$$- \tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^5\gamma^\mu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\beta_d\right) - \tfrac{i}{8}\left(\sigma^{\lambda\varphi}\right)_{ab}\left(\gamma^5\gamma^\mu\right)_{ce}\left(\sigma_{\lambda\varphi}\right)^{ef}\left(\gamma^5\right)_f{}^d\left(\partial_\mu\beta_d\right)$$
$$- \tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\nu\gamma^\mu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma_\nu\right)_f{}^d(\partial_\mu\beta_d) - \tfrac{i}{8}(\sigma^{\lambda\varphi}_{ab}(\gamma^\nu\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma_\nu)_f{}^d(\partial_\mu\beta_d)$$
$$- \tfrac{i}{8}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\gamma^\nu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma_\nu\right)_f{}^d(\partial_\mu\beta_d) - \tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^\mu\gamma^\nu)_{ce}(\sigma_{\lambda[10pt]\varphi})^{ef}(\gamma_\nu)_f{}^d(\partial_\mu\beta_d)$$
$$+ \tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma^5\gamma_\nu\right)_f{}^d(\partial_\mu\beta_d) + \tfrac{i}{8}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\nu\gamma^\mu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma^5\gamma_\nu)_f{}^d(\partial_\mu\beta_d)$$
$$+ \tfrac{i}{8}\left(\gamma^\lambda\right)_{ab}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_{ce}\left(\gamma_\lambda\right)^{ef}\left(\gamma^5\gamma_\nu\right)_f{}^d(\partial_\mu\beta_d) + \tfrac{i}{16}(\sigma^{\lambda\varphi})_{ab}(\gamma^5\gamma^\mu\gamma^\nu)_{ce}(\sigma_{\lambda\varphi})^{ef}(\gamma^5\gamma_\nu)_f{}^d(\partial_\mu\beta_d)$$
$$= \tfrac{i}{2}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\gamma_\lambda\right)_c{}^d\left(\partial_\mu\beta_d\right) + \tfrac{i}{2}(\gamma^\lambda)_{ab}(\gamma^\nu\gamma^\mu\gamma_\lambda\gamma_\nu)_c{}^d(\partial_\mu\beta_d) + \tfrac{i}{4}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma^\nu\gamma_\lambda\gamma_\nu)_c{}^d(\partial_\mu\beta_d)$$
$$= \tfrac{i}{2}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\gamma_\lambda\right)_c{}^d\left(\partial_\mu\beta_d\right) + \tfrac{i}{2}\eta^{\mu\varphi}\left(\gamma^\lambda\right)_{ab}\left(4\eta_{\varphi\lambda}\mathbf{I}\right)_c{}^d\left(\partial_\mu\beta_d\right) + \tfrac{i}{4}\left(\gamma^\lambda\right)_{ab}\left(\gamma^\mu\left(-2\gamma_\lambda\right)\right)_c{}^d\left(\partial_\mu\beta_d\right)$$
$$= \tfrac{i}{2}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda)_c{}^d(\partial_\mu\beta_d) + 2i\delta_\lambda{}^\mu\delta_c{}^d(\gamma^\lambda)_{ab}(\partial_\mu\beta_d) - \tfrac{i}{2}(\gamma^\lambda)_{ab}(\gamma^\mu\gamma_\lambda)_c{}^d(\partial_\mu\beta_d)$$
$$= 2i\left(\gamma^\mu\right)_{ab}\left(\partial_\mu\beta_c\right) \tag{73}$$

$$\{D_a, D_b\}\beta_c = 2i\left(\gamma^\mu\right)_{ab}\left(\partial_\mu\beta_c\right) \tag{74}$$

2 Invariance of the Lagrangian

In this section we calculate the Lagrangian invariance under the supersymmetry covariant derivation. [1]

$$
\begin{aligned}
&D_a\mathcal{L} = -\tfrac{1}{2}D_a(\partial_\mu K\partial^\mu K) - \tfrac{1}{2}(D_a(\partial_\mu L\partial^\mu L)\\
&-\tfrac{1}{2}D_a(M^2) - \tfrac{1}{2}D_a(N^2) + \tfrac{1}{4}D_a(V_mV^m) + \tfrac{1}{4}D_a(U_mU^m)\\
&+\tfrac{i}{2}(\gamma^m)^{bc}D_a(\zeta_b\partial_\mu\zeta_c) + iC^{bc}D_a(\rho_b\beta_c)\\
&= A_a + B_a + C_a + E_a + F_a + G_a + H_a + J_a
\end{aligned} \tag{75}
$$

2.1 Bosonic Field

$$
\begin{aligned}
&A_a = -\tfrac{1}{2}(\partial_\mu D_aK)(\partial^\mu K) - \tfrac{1}{2}(\partial_\mu K)(\partial^\mu D_aK)\\
&= -((\partial_\mu\rho_a) - (\partial_\mu\zeta_a))(\partial^\mu K)\\
&= -(\partial_\mu\rho_a)(\partial^\mu K) + (\partial_\mu\zeta_a)(\partial^\mu K)
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
&B_a = -\tfrac{1}{2}(\partial_\mu D_aL)(\partial^\mu L) - \tfrac{1}{2}(\partial^\mu D_aL)(\partial_\mu L)\\
&= -(-i(\gamma^5)_a{}^b(\partial_\mu\rho_b) - i(\gamma^5)_a{}^b(\partial_\mu\zeta_\beta))(\partial^\mu L)\\
&= i(\gamma^5)_a{}^b(\partial_\mu\rho_b)(\partial^\mu L) + i(\gamma^5)_a{}^b(\partial_\mu\zeta_b)(\partial^\mu L)
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
&C_a = -M(D_aM) = -M(\beta_a - \tfrac{1}{2}(\gamma^\mu)_a{}^b(\partial_\mu\rho_b))\\
&= -M\beta_a + \tfrac{1}{2}(\gamma^\mu)_a{}^bM(\partial_\mu\rho_b)
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
&E_a = -N\left(D_aN\right) = -N\left(-i\left(\gamma^5\right)_a{}^b\beta_b + \tfrac{i}{2}\left(\gamma^5\gamma^\mu\right)_a{}^b\left(\partial_\mu\rho_b\right)\right)\\
&= i\left(\gamma^5\right)_a{}^bN\beta_b - \tfrac{i}{2}\left(\gamma^5\gamma^\mu\right)_a{}^bN\left(\partial_\mu\rho_b\right)
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
&F_a = \tfrac{1}{4}(D_aV_\mu)V^\mu + \tfrac{1}{4}V_\mu(D_aV^\mu)\\
&= \tfrac{1}{2}(-(\gamma_\mu)_a{}^b\beta_b - (\gamma_\mu\gamma^\nu)_a{}^b(\partial_\nu\zeta_b) + (\partial_\mu\rho_\alpha) + \tfrac{1}{2}(\gamma^\nu\gamma_\mu)_a{}^b(\partial_\nu\rho_b))V^\mu\\
&= -\tfrac{1}{2}\left(\gamma_\mu\right)_a{}^b\beta_bV^\mu - \tfrac{1}{2}\left(\gamma_\mu\gamma^\nu\right)_a{}^b\left(\partial_\nu\zeta_b\right)V^\mu + \tfrac{1}{2}(\partial_\mu\rho_\alpha)V^\mu + \tfrac{1}{4}\left(\gamma^\nu\gamma_\mu\right)_a{}^b\left(\partial_\nu\rho_b\right)V^\mu
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
&G_a = \tfrac{1}{4}\left(D_a U_\mu\right) U^\mu + \tfrac{1}{4} U_\mu \left(D_a U^\mu\right) \\
&= \tfrac{1}{2}(i \left(\gamma^5 \gamma_\mu\right)_a{}^b \beta_b - i\left(\gamma^5 \gamma_\mu \gamma^\nu\right)_a{}^b \left(\partial_\nu \zeta_b\right) - i\left(\gamma^5\right)_a{}^b \left(\partial_\mu \rho_b\right) - \tfrac{i}{2}\left(\gamma^5 \gamma^\nu \gamma_\mu\right)_a{}^b \left(\partial_\nu \rho_b\right) U^\mu \\
&= \tfrac{i}{2}\left(\gamma^5 \gamma_\mu\right)_a{}^b \beta_b U^\mu - \tfrac{i}{2}\left(\gamma^5 \gamma_\mu \gamma^\nu\right)_a{}^b \left(\partial_\nu \zeta_b\right) U^\mu \\
&- \tfrac{i}{2}\left(\gamma^5\right)_a{}^b \left(\partial_\mu \rho_b\right) U^\mu - \tfrac{i}{4}\left(\gamma^5 \gamma^\nu \gamma_\mu\right)_a{}^b \left(\partial_\nu \rho_b\right) U^\mu
\end{aligned}
\tag{81}
$$

2.2 Fermionic Field

$$
\begin{aligned}
&H_a = \tfrac{i}{2}\left(\gamma^\mu\right)^{bc}\left(D_\alpha \zeta_b\right)\left(\partial_\mu \zeta_c\right) - \tfrac{i}{2}\left(\gamma^\mu\right)^{bc} \zeta_b \left(\partial_\mu D_a \zeta_c\right) \\
&= \tfrac{i}{2}\left(\gamma^\mu\right)^{bc}\left\{-i\left(\gamma^\nu\right)_{ab}\left(\partial_\nu K\right) + \left(\gamma^5 \gamma^\nu\right)_{ab}\left(\partial_\nu L\right) + \tfrac{i}{2}\left(\gamma^\nu\right)_{ab} V_\nu - \tfrac{1}{2}\left(\gamma^5 \gamma^\nu\right)_{ab} U_\nu\right\}\left(\partial_\mu \zeta_c\right) \\
&- \tfrac{i}{2}\left(\gamma^\mu\right)^{bc} \zeta_b \partial_\mu \left\{-i\left(\gamma^\nu\right)_{ac}\left(\partial_\nu K\right) + \left(\gamma^5 \gamma^\nu\right)_{ac}\left(\partial_\nu L\right) + \tfrac{i}{2}\left(\gamma^\nu\right)_{ac} V_\nu - \tfrac{1}{2}\left(\gamma^5 \gamma^\nu\right)_{ac} U_\nu\right\} \\
&= -\left(\gamma^\nu \gamma^\mu\right)_a{}^b \left(\partial_\nu K\right)\left(\partial_\mu \zeta_b\right) - i\left(\gamma^5 \gamma^\nu \gamma^\mu\right)_a{}^b \left(\partial_\nu L\right)\left(\partial_\mu \zeta_b\right) + \tfrac{1}{2}\left(\gamma^\nu \gamma^\mu\right)_a{}^b V_\nu \left(\partial_\mu \zeta_b\right) \\
&+ \tfrac{i}{2}\left(\gamma^5 \gamma^\nu \gamma^\mu\right)_a{}^b U_\nu \left(\partial_\mu \zeta_b\right)
\end{aligned}
\tag{82}
$$

And let us divide J field in two part,

$$
J_a = iC^{bc}\left(D_a \rho_b\right)\beta_c - iC^{bc}\rho_b\left(D_a \beta_c\right) = J_a{}^1 + J_a{}^2 \tag{83}
$$

$$
\begin{aligned}
&J_a{}^1 = iC^{bc}\left\{iC_{ab}M + \left(\gamma^5\right)_{ab} N + \tfrac{i}{2}\left(\gamma^\mu\right)_{ab} V_\mu + \tfrac{1}{2}\left(\gamma^5 \gamma^\mu\right)_{ab} U_\mu\right\}\beta_c \\
&= \delta_a{}^c M \beta_c - i\left(\gamma^5\right)_a{}^c N \beta_c + \tfrac{1}{2}\left(\gamma^\mu\right)_a{}^c V_\mu \beta_c - \tfrac{i}{2}\left(\gamma^5 \gamma^\mu\right)_a{}^c U_\mu \beta_c
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&J_a{}^2 = -iC^{bc}\rho_b\left\{-\eta^{\mu\nu}\partial_\mu \partial_\nu \left(iC_{ac}K + \left(\gamma^5\right)_{ac} L\right) + \tfrac{i}{2}\left(\gamma^\mu\right)_{ac}\left(\partial_\mu M\right) + \tfrac{1}{2}\left(\gamma^5 \gamma^\mu\right)_{ac}\left(\partial_\mu N\right)\right. \\
&\left. + \tfrac{i}{2}(\gamma^\mu \gamma^\nu)_{ac}(\partial_\mu V_\nu) + \tfrac{i}{4}(\gamma^\nu \gamma^\mu)_{ac}(\partial_\mu V_\nu) + \tfrac{1}{2}(\gamma^5 \gamma^\mu \gamma^\nu)_{ac}(\partial_\mu U_\nu) + \tfrac{1}{4}(\gamma^5 \gamma^\nu \gamma^\mu)_{ac}(\partial_\mu U_\nu)\right\} \\
&= -\eta^{\mu\nu}\delta_a{}^b \rho_b \left(\partial_\mu \partial_\nu K\right) + i\eta^{\mu\nu}\left(\gamma^5\right)_a{}^b \rho_b \left(\partial_\mu \partial_\nu L\right) + \tfrac{1}{2}\left(\gamma^\mu\right)_a{}^b \rho_b \left(\partial_\mu M\right) \\
&- \tfrac{i}{2}\left(\gamma^5 \gamma^\mu\right)_a{}^b \rho_b \left(\partial_\mu N\right) + \tfrac{1}{2}(\gamma^\mu \gamma^\nu)_a{}^b \rho_b (\partial_\mu V_\nu) + \tfrac{1}{4}(\gamma^\nu \gamma^\mu)_a{}^b \rho_b (\partial_\mu V_\nu) \\
&- \tfrac{i}{2}(\gamma^5 \gamma^\mu \gamma^\nu)_a{}^b \rho_b (\partial_\mu U_\nu) - \tfrac{i}{4}(\gamma^5 \gamma^\nu \gamma^\mu)_a{}^b \rho_b (\partial_\mu U_\nu)
\end{aligned}
\tag{85}
$$

2.3 Sum up and matching terms

K/ζ terms:

$$\begin{aligned}&(\partial_\mu \zeta_\alpha)(\partial^\mu K) - (\gamma^\nu \gamma^\mu)_a{}^b (\partial_\nu K)(\partial_\mu \zeta_b) = \{\eta^{\mu\nu}\mathbf{I} - \gamma^\nu\gamma^\mu\}_a{}^b (\partial_\nu K)(\partial_\mu \zeta_b)\\ &= \tfrac{1}{2}\{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\}_a^b (\partial_\nu K)(\partial_\mu \zeta_b) = -i(\sigma^{\mu\nu})_a{}^b (\partial_\nu K)(\partial_\mu \zeta_b)\\ &= i\,(\sigma^{\mu\nu})_a{}^b K(\partial_\mu\partial_\nu\zeta_b) = i\,(\sigma^{\nu\mu})_a{}^b K(\partial_\nu\partial_\mu\zeta_b) = -i\,(\sigma^{\mu\nu})_a{}^b K(\partial_\mu\partial_\nu\zeta_b) = 0\end{aligned} \tag{86}$$

(negative of self)

L/ζ terms:

$$\begin{aligned}&i(\gamma^5)_a{}^b(\partial_\mu\zeta_b)(\partial^\mu L) - i(\gamma^5\gamma^\nu\gamma^\mu)_a{}^b(\partial_\nu L)(\partial_\mu\zeta_b)\\ &= i\{\gamma^5(\eta^{\mu\nu}\mathbf{I} - \gamma^\nu\gamma^\mu)\}_a{}^b(\partial_\nu L)(\partial_\mu\zeta_b)\\ &= \tfrac{i}{2}\left\{\gamma^5(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right\}_a{}^b(\partial_\nu L)(\partial_\mu\zeta_b) = \left(\gamma^5\sigma^{\mu\nu}\right)_a{}^b(\partial_\nu L)(\partial_\mu\zeta_b)\\ &= -\left(\gamma^5\sigma^{\nu\mu}\right)_a{}^b L(\partial_\nu\partial_\mu\zeta_b) = \left(\gamma^5\sigma^{\mu\nu}\right)_a{}^b L(\partial_\mu\partial_\nu\zeta_b) = 0\end{aligned} \tag{87}$$

(negative of self)

V/ζ firms:

$$\begin{aligned}&-\tfrac{1}{2}(\gamma_\mu\gamma^\nu)_a{}^b(\partial_\nu\zeta_b)V^\mu + \tfrac{1}{2}(\gamma^\nu\gamma^\mu)_a{}^b(\partial_\mu\zeta_b)V_\nu\\ &= -\tfrac{1}{2}(\gamma^\mu\gamma^\nu)_a{}^b(\partial_\nu\zeta_b)V_\mu + \tfrac{1}{2}(\gamma^\mu\gamma^\nu)_a{}^b(\partial_\nu\zeta_b)V_\mu = 0\end{aligned} \tag{88}$$

U/ζ terms:

$$\begin{aligned}&-\tfrac{i}{2}\left(\gamma^5\gamma_\mu\gamma^\nu\right)_a{}^b(\partial_\nu\zeta_b)U^\mu + \tfrac{i}{2}\left(\gamma^5\gamma^\nu\gamma^\mu\right)_a{}^b(\partial_\mu\zeta_b)U_\nu\\ &= -\tfrac{i}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_a{}^b(\partial_\nu\zeta_b)U_\mu + \tfrac{i}{2}\left(\gamma^5\gamma^\mu\gamma^\nu\right)_a{}^b(\partial_\mu\zeta_b)U_\mu = 0\end{aligned} \tag{89}$$

K/ρ terms:

$$-(\partial_\mu\rho_a)(\partial^\mu K) - \eta^{\mu\nu}\delta_a{}^b\rho_b(\partial_\mu\partial_\nu K) = -(\partial_\mu\rho_a)(\partial^\mu K) + (\partial_\mu\rho_a)(\partial^\mu K) = 0 \tag{90}$$

L/ρ terms:

$$\begin{aligned}&i(\gamma^5)_a{}^b(\partial_\mu\rho_b)(\partial^\mu L) + i\eta^{\mu\nu}(\gamma^5)_a{}^b\rho_b(\partial_\mu\partial_\nu L)\\ &= i(\gamma^5)_a{}^b(\partial_\mu\rho_b)(\partial^\mu L) - i(\gamma^5)_a{}^b(\partial_\mu\rho_b)(\partial^\mu L) = 0\end{aligned} \tag{91}$$

M/ρ terms:

$$\frac{1}{2}\left(\gamma^{\mu}\right)_{a}{}^{b} M\left(\partial_{\mu}\rho_{b}\right)+\frac{1}{2}\left(\gamma^{\mu}\right)_{a}{}^{b}\rho_{b}\left(\partial_{\mu}M\right)$$
$$=\frac{1}{2}\left(\gamma^{\mu}\right)_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)M-\frac{1}{2}\left(\gamma^{\mu}\right)_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)M=0 \tag{92}$$

N/ρ terms:

$$-\frac{i}{2}(\gamma^{5}\gamma^{\mu})_{a}{}^{b}N(\partial_{\mu}\rho_{b})-\frac{i}{2}(\gamma^{5}\gamma^{\mu})_{a}{}^{b}\rho_{b}(\partial_{\mu}N)$$
$$=-\frac{i}{2}(\gamma^{5}\gamma^{\mu})_{a}{}^{b}(\partial_{\mu}\rho_{b})N+\frac{i}{2}(\gamma^{5}\gamma^{m})_{a}{}^{b}(\partial_{\mu}\rho_{b})N=0 \tag{93}$$

V/ρ terms:

$$\frac{1}{2}\left(\partial_{\mu}\rho_{b}\right)V^{\mu}+\frac{1}{4}\left(\gamma^{\nu}\gamma_{\mu}\right)_{a}{}^{b}\left(\partial_{\nu}\rho_{b}\right)V^{\mu}+\frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu}\right)_{a}{}^{b}\rho_{b}\left(\partial_{\mu}V_{\nu}\right)+\frac{1}{4}\left(\gamma^{\nu}\gamma^{\mu}\right)_{a}{}^{b}\rho_{b}\left(\partial_{\mu}V_{\nu}\right)$$
$$=\frac{1}{2}\left\{\eta^{\mu\nu}\mathbf{I}-\gamma^{\mu}\gamma^{\nu}\right\}_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)V_{\nu}+\frac{1}{4}\left(\gamma^{\mu}\gamma^{\nu}\right)_{a}^{b}\left(\partial_{\mu}\rho_{b}\right)V_{\nu}-\frac{1}{4}\left(\gamma^{\nu}\gamma^{\mu}\right)_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)V_{\nu}$$
$$=\frac{1}{4}\left\{\gamma^{\nu}\gamma^{\mu}-\gamma^{\mu}\gamma^{\nu}\right\}_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)V_{\nu}+\frac{1}{4}\left\{\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}\right\}_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)V_{\nu}=0 \tag{94}$$

U/ρ terms:

$$-\frac{i}{2}\left(\gamma^{5}\right)_{a}{}^{b}\left(\partial_{\mu}\rho_{b}\right)U^{\mu}-\frac{i}{4}\left(\gamma^{5}\gamma^{\nu}\gamma_{\mu}\right)_{a}{}^{b}\left(\partial_{\nu}\rho_{b}\right)U^{\mu}-\frac{i}{2}\left(\gamma^{5}\gamma^{\mu}\gamma^{\nu}\right)_{a}{}^{b}\rho_{b}\left(\partial_{\mu}U_{\nu}\right)$$
$$-\frac{i}{4}\left(\gamma^{5}\gamma^{\nu}\gamma^{\mu}\right)_{a}{}^{b}\rho_{b}\left(\partial_{\mu}U_{\nu}\right)$$
$$=-\frac{i}{2}\left\{\gamma^{5}(\eta^{\mu\nu}\mathbf{I}-\gamma^{\mu}\gamma^{\nu})\right\}_{a}{}^{b}(\partial_{\mu}\rho_{b})U_{\nu}-\frac{i}{4}(\gamma^{5}\gamma^{\mu}\gamma^{\nu})_{a}{}^{b}(\partial_{\mu}\rho_{b})U_{\nu}$$
$$+\frac{i}{4}(\gamma^{5}\gamma^{\nu}\gamma^{\mu})_{a}{}^{b}(\partial_{\mu}\rho_{b})U_{\nu}$$
$$=-\frac{i}{4}\{\gamma^{5}\gamma^{\nu}\gamma^{\mu}-\gamma^{5}\gamma^{\mu}\gamma^{\nu}\}_{a}{}^{b}(\partial_{\mu}\rho_{b})U_{\nu}+\frac{i}{4}\{\gamma^{5}\gamma^{\nu}\gamma^{\mu}-\gamma^{5}\gamma^{\mu}\gamma^{\nu}\}_{a}{}^{b}(\partial_{\mu}\rho_{b})U_{\nu}=0 \tag{95}$$

M/β terms:

$$-M\beta_{a}+\delta_{a}{}^{c}M\beta_{c}=-M\beta_{a}+M\beta_{a}=0 \tag{96}$$

N/β terms:

$$i\left(\gamma^{5}\right)_{a}{}^{b}N\beta_{b}-i\left(\gamma^{5}\right)_{a}{}^{c}N\beta_{c}=0 \tag{97}$$

V/β terms:

$$-\frac{1}{2}\left(\gamma_{\mu}\right)_{a}{}^{b}\beta_{b}V^{\mu}+\frac{1}{2}\left(\gamma^{\mu}\right)_{a}{}^{c}V_{\mu}\beta_{c}=0 \tag{98}$$

U/β terms:

$$\frac{i}{2}\left(\gamma^{5}\gamma_{\mu}\right)_{a}{}^{b}\beta_{b}U^{\mu}-\frac{i}{2}\left(\gamma^{5}\gamma^{\mu}\right)_{a}{}^{c}U_{\mu}\beta_{c}=0 \tag{99}$$

Therefore,

$$D_{a}\mathcal{L}=0+\partial^{\mu}\mathcal{J}_{\mu a} \tag{100}$$

3 Adinkra Calculations

In this section we calculate the Adinkra for CLS [1, 2, 3]. To verify the element in L-matrix as 1 or -1, we need to mix the bosonic and fermionic field as linear combination.

3.1 Original Field Transformations

$$
\begin{aligned}
D_1 K &= \rho_1 - \zeta_1 \qquad D_2 K = \rho_2 - \zeta_2 \\
D_3 K &= \rho_3 - \zeta_3 \qquad D_4 K = \rho_4 - \zeta_4
\end{aligned} \tag{101}
$$

$$
\begin{aligned}
D_1 L &= \rho_4 + \zeta_4 \qquad D_2 L = -\rho_3 - \zeta_3 \\
D_3 L &= \rho_2 + \zeta_2 \qquad D_4 L = -\rho_1 - \zeta_1
\end{aligned} \tag{102}
$$

$$
\begin{aligned}
D_1 M &= \beta_1 - \frac{1}{2}\dot{\rho}_2 \qquad D_2 M = \beta_2 + \frac{1}{2}\dot{\rho}_1 \\
D_3 M &= \beta_3 + \frac{1}{2}\dot{\rho}_4 \qquad D_4 M = \beta_4 - \frac{1}{2}\dot{\rho}_3
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
D_1 N &= \beta_4 - \frac{1}{2}\dot{\rho}_3 \qquad D_2 N = -\beta_3 - \frac{1}{2}\dot{\rho}_4 \\
D_3 N &= \beta_2 + \frac{1}{2}\dot{\rho}_1 \qquad D_4 N = -\beta_1 + \frac{1}{2}\dot{\rho}_2
\end{aligned} \tag{104}
$$

$$
\begin{aligned}
D_1 V_0 &= \beta_2 - \dot{\zeta}_1 + \frac{3}{2}\dot{\rho}_1 \qquad D_2 V_0 = -\beta_1 - \dot{\zeta}_2 + \frac{3}{2}\dot{\rho}_2 \\
D_3 V_0 &= -\beta_4 - \dot{\zeta}_3 + \frac{3}{2}\dot{\rho}_3 \qquad D_4 V_0 = \beta_3 - \dot{\zeta}_4 + \frac{3}{2}\dot{\rho}_4
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
D_1 V_1 &= -\beta_2 + \dot{\zeta}_1 + \frac{1}{2}\dot{\rho}_1 \qquad D_2 V_1 = -\beta_1 - \dot{\zeta}_2 - \frac{1}{2}\dot{\rho}_2 \\
D_3 V_1 &= -\beta_4 - \dot{\zeta}_3 - \frac{1}{2}\dot{\rho}_3 \qquad D_4 V_1 = -\beta_3 + \dot{\zeta}_4 + \frac{1}{2}\dot{\rho}_4
\end{aligned} \tag{106}
$$

$$
\begin{aligned}
D_1 V_2 &= \beta_4 + \dot{\zeta}_3 + \frac{1}{2}\dot{\rho}_3 \qquad D_2 V_2 = -\beta_3 + \dot{\zeta}_4 + \frac{1}{2}\dot{\rho}_4 \\
D_3 V_2 &= -\beta_2 + \dot{\zeta}_1 + \frac{1}{2}\dot{\rho}_1 \qquad D_4 V_2 = \beta_1 + \dot{\zeta}_2 + \frac{1}{2}\dot{\rho}_2
\end{aligned} \tag{107}
$$

$$D_1V_3 = -\beta_1 - \dot\zeta_2 - \frac{1}{2}\dot\rho_2 \qquad D_2V_3 = \beta_2 - \dot\zeta_1 - \frac{1}{2}\dot\rho_1$$
$$D_3V_3 = -\beta_3 + \dot\zeta_4 + \frac{1}{2}\dot\rho_4 \qquad D_4V_3 = \beta_4 + \dot\zeta_3 + \frac{1}{2}\dot\rho_3 \tag{108}$$

$$D_1U_0 = \beta_3 + \dot\zeta_4 + \frac{3}{2}\dot\rho_4 \qquad D_2U_0 = \beta_4 - \dot\zeta_3 - \frac{3}{2}\dot\rho_3$$
$$D_3U_0 = -\beta_1 + \dot\zeta_2 + \frac{3}{2}\dot\rho_2 \qquad D_4U_0 = -\beta_2 - \dot\zeta_1 - \frac{3}{2}\dot\rho_1 \tag{109}$$

$$D_1U_1 = -\beta_3 - \dot\zeta_4 + \frac{1}{2}\dot\rho_4 \qquad D_2U_1 = \beta_4 - \dot\zeta_3 + \frac{1}{2}\dot\rho_3$$
$$D_3U_1 = -\beta_1 + \dot\zeta_2 - \frac{1}{2}\dot\rho_2 \qquad D_4U_1 = \beta_2 + \dot\zeta_1 - \frac{1}{2}\dot\rho_1 \tag{110}$$

$$D_1U_2 = \beta_1 - \dot\zeta_2 + \frac{1}{2}\dot\rho_2 \qquad D_2U_2 = \beta_2 + \dot\zeta_1 - \frac{1}{2}\dot\rho_1$$
$$D_3U_2 = -\beta_3 - \dot\zeta_4 + \frac{1}{2}\dot\rho_4 \qquad D_4U_2 = -\beta_4 + \dot\zeta_3 - \frac{1}{2}\dot\rho_3 \tag{111}$$

$$D_1U_3 = \beta_4 - \dot\zeta_3 + \frac{1}{2}\dot\rho_3 \qquad D_2U_3 = \beta_3 + \dot\zeta_4 - \frac{1}{2}\dot\rho_4$$
$$D_3U_3 = \beta_2 + \dot\zeta_1 - \frac{1}{2}\dot\rho_1 \qquad D_4U_3 = \beta_1 - \dot\zeta_2 + \frac{1}{2}\dot\rho_2 \tag{112}$$

$$D_1\zeta_1 = -i\dot K + \frac{i}{2}V_0 + \frac{i}{2}V_1 \qquad D_2\zeta_1 = -\frac{i}{2}V_3 + \frac{i}{2}U_2$$
$$D_3\zeta_1 = \frac{i}{2}V_2 + \frac{i}{2}U_3 \qquad D_4\zeta_1 = -i\dot L + \frac{i}{2}U_0 + \frac{i}{2}U_1 \tag{113}$$

$$D_1\zeta_2 = -\frac{i}{2}V_3 - \frac{i}{2}U_2 \qquad D_2\zeta_2 = -i\dot K + \frac{i}{2}V_0 - \frac{i}{2}V_1$$
$$D_3\zeta_2 = i\dot L - \frac{i}{2}U_0 + \frac{i}{2}U_1 \qquad D_4\zeta_2 = \frac{i}{2}V_2 - \frac{i}{2}U_3 \tag{114}$$

$$D_1\zeta_3 = \frac{i}{2}V_2 - \frac{i}{2}U_3 \qquad D_2\zeta_3 = -i\dot L + \frac{i}{2}U_0 - \frac{i}{2}U_1$$
$$D_3\zeta_3 = -i\dot K + \frac{i}{2}V_0 - \frac{i}{2}V_1 \qquad D_4\zeta_3 = \frac{i}{2}V_3 + \frac{i}{2}U_2 \tag{115}$$

$$D_1\zeta_4 = i\dot L - \frac{i}{2}U_0 - \frac{i}{2}U_1 \qquad D_2\zeta_4 = \frac{i}{2}V_2 + \frac{i}{2}U_3$$
$$D_3\zeta_4 = \frac{i}{2}V_3 - \frac{i}{2}U_2 \qquad D_4\zeta_4 = -i\dot K + \frac{i}{2}V_0 + \frac{i}{2}V_1 \tag{116}$$

$$D_1\rho_1 = \frac{i}{2}V_0 + \frac{i}{2}V_1 \qquad D_2\rho_1 = iM - \frac{i}{2}V_3 - \frac{i}{2}U_2$$
$$D_3\rho_1 = iN + \frac{i}{2}V_2 - \frac{i}{2}U_3 \qquad D_4\rho_1 = -\frac{i}{2}U_0 - \frac{i}{2}U_1 \tag{117}$$

$$D_1\rho_2 = -iM - \frac{i}{2}V_3 + \frac{i}{2}U_2 \qquad D_2\rho_2 = \frac{i}{2}V_0 - \frac{i}{2}V_1$$
$$D_3\rho_2 = \frac{i}{2}U_0 - \frac{i}{2}U_1 \qquad D_4\rho_2 = iN + \frac{i}{2}V_2 + \frac{i}{2}U_3 \tag{118}$$

$$D_1\rho_3 = -iN + \frac{i}{2}V_2 + \frac{i}{2}U_3 \qquad D_2\rho_3 = -\frac{i}{2}U_0 + \frac{i}{2}U_1$$
$$D_3\rho_3 = \frac{i}{2}V_0 - \frac{i}{2}V_1 \qquad D_4\rho_3 = -iM + \frac{i}{2}V_3 - \frac{i}{2}U_2 \tag{119}$$

$$D_1\rho_4 = \frac{i}{2}U_0 + \frac{i}{2}U_1 \qquad D_2\rho_4 = -iN + \frac{i}{2}V_2 - \frac{i}{2}U_3$$
$$D_3\rho_4 = iM + \frac{i}{2}V_3 + \frac{i}{2}U_2 \qquad D_4\rho_4 = \frac{i}{2}V_0 + \frac{i}{2}V_1 \tag{120}$$

$$D_1\beta_1 = \frac{i}{2}\dot{M} - \frac{i}{4}\dot{V}_3 + \frac{i}{4}\dot{U}_2 \qquad D_2\beta_1 = i\ddot{K} - \frac{3i}{4}\dot{V}_0 - \frac{i}{4}\dot{V}_1$$
$$D_3\beta_1 = i\ddot{L} - \frac{3i}{4}\dot{U}_0 - \frac{i}{4}\dot{U}_1 \qquad D_4\beta_1 = -\frac{i}{2}\dot{N} + \frac{i}{4}\dot{V}_2 + \frac{i}{4}\dot{U}_3 \tag{121}$$

$$D_1\beta_2 = -i\ddot{K} + \frac{3i}{4}\dot{V}_0 - \frac{i}{4}\dot{V}_1 \qquad D_2\beta_2 = \frac{i}{2}\dot{M} + \frac{i}{4}\dot{V}_3 + \frac{i}{4}\dot{U}_2$$
$$D_3\beta_2 = \frac{i}{2}\dot{N} - \frac{i}{4}\dot{V}_2 + \frac{i}{4}\dot{U}_3 \qquad D_4\beta_2 = i\ddot{L} - \frac{3i}{4}\dot{U}_0 + \frac{i}{4}\dot{U}_1 \tag{122}$$

$$D_1\beta_3 = -i\ddot{L} + \frac{3i}{4}\dot{U}_0 - \frac{i}{4}\dot{U}_1 \qquad D_2\beta_3 = -\frac{i}{2}\dot{N} - \frac{i}{4}\dot{V}_2 + \frac{i}{4}\dot{U}_3$$
$$D_3\beta_3 = \frac{i}{2}\dot{M} - \frac{i}{4}\dot{V}_3 - \frac{i}{4}\dot{U}_2 \qquad D_4\beta_3 = -i\ddot{K} + \frac{3i}{4}\dot{V}_0 - \frac{i}{4}\dot{V}_1 \tag{123}$$

$$D_1\beta_4 = \frac{i}{2}\dot{N} + \frac{i}{4}\dot{V}_2 + \frac{i}{4}\dot{U}_3 \qquad D_2\beta_4 = -i\ddot{L} + \frac{3i}{4}\dot{U}_0 + \frac{i}{4}\dot{U}_1$$
$$D_3\beta_4 = i\ddot{K} - \frac{3i}{4}\dot{V}_0 - \frac{i}{4}\dot{V}_1 \qquad D_4\beta_4 = \frac{i}{2}\dot{M} + \frac{i}{4}\dot{V}_3 - \frac{i}{4}\dot{U}_2 \tag{124}$$

3.2 New Field Definitions

$$\begin{array}{llll}
\dot{\Phi}_1 = M & \dot{\Phi}_2 = V_0 - \dot{K} & \dot{\Phi}_3 = U_0 - \dot{L} & \dot{\Phi}_4 = N \\
\dot{\Phi}_5 = U_2 & \dot{\Phi}_6 = V_0 - 2\dot{K} & \dot{\Phi}_7 = -U_1 & \dot{\Phi}_8 = U_3 \\
\dot{\Phi}_9 = -V_3 & \dot{\Phi}_{10} = -V_1 & \dot{\Phi}_{11} = U_0 - 2\dot{L} & \dot{\Phi}_{12} = V_2
\end{array} \tag{125}$$

$$\begin{array}{ll}
i\dot{\Psi}_1 = \beta_1 - \frac{1}{2}\dot{\rho}_2 & i\dot{\Psi}_2 = \beta_2 + \frac{1}{2}\dot{\rho}_1 \\
i\dot{\Psi}_3 = \beta_3 + \frac{1}{2}\dot{\rho}_4 & i\dot{\Psi}_4 = \beta_4 - \frac{1}{2}\dot{\rho}_3 \\
i\dot{\Psi}_5 = \beta_1 - \dot{\zeta}_2 + \frac{1}{2}\dot{\rho}_2 & i\dot{\Psi}_6 = \beta_2 + \dot{\zeta}_1 - \frac{1}{2}\dot{\rho}_1 \\
i\dot{\Psi}_7 = \beta_3 + \dot{\zeta}_4 - \frac{1}{2}\dot{\rho}_4 & i\dot{\Psi}_8 = \beta_4 - \dot{\zeta}_3 + \frac{1}{2}\dot{\rho}_3 \\
i\dot{\Psi}_9 = \beta_1 + \dot{\zeta}_2 + \frac{1}{2}\dot{\rho}_2 & i\dot{\Psi}_{10} = \beta_2 - \dot{\zeta}_1 - \frac{1}{2}\dot{\rho}_1 \\
i\dot{\Psi}_{11} = \beta_3 - \dot{\zeta}_4 - \frac{1}{2}\dot{\rho}_4 & i\dot{\Psi}_{12} = \beta_4 + \dot{\zeta}_3 + \frac{1}{2}\dot{\rho}_3
\end{array} \tag{126}$$

3.3 New Field Transformations

First for D_1,

$$\begin{array}{ll}
D_1(\dot{\Phi}_1) = \beta_1 - \frac{1}{2}\dot{\rho}_2 = i\dot{\Psi}_1 & D_1(\dot{\Phi}_2) = \beta_2 + \frac{1}{2}\dot{\rho}_1 = i\dot{\Psi}_2 \\
D_1(\dot{\Phi}_3) = \beta_3 + \frac{1}{2}\dot{\rho}_4 = i\dot{\Psi}_3 & D_1(\dot{\Phi}_4) = \beta_4 - \frac{1}{2}\dot{\rho}_3 = i\dot{\Psi}_4
\end{array} \tag{127}$$

$$\begin{array}{ll}
D_1(\dot{\Phi}_5) = \beta_1 - \dot{\zeta}_2 + \frac{1}{2}\dot{\rho}_2 = i\dot{\Psi}_5 & D_1(\dot{\Phi}_6) = \beta_2 + \dot{\zeta}_1 - \frac{1}{2}\dot{\rho}_1 = i\dot{\Psi}_6 \\
D_1(\dot{\Phi}_7) = \beta_3 + \dot{\zeta}_4 - \frac{1}{2}\dot{\rho}_4 = i\dot{\Psi}_7 & D_1(\dot{\Phi}_8) = \beta_4 - \dot{\zeta}_3 + \frac{1}{2}\dot{\rho}_3 = i\dot{\Psi}_8
\end{array} \tag{128}$$

$$\begin{array}{ll}
D_1(\dot{\Phi}_9) = \beta_1 + \dot{\zeta}_2 + \frac{1}{2}\dot{\rho}_2 = i\dot{\Psi}_9 & D_1(\dot{\Phi}_{10}) = \beta_2 - \dot{\zeta}_1 - \frac{1}{2}\dot{\rho}_1 = i\dot{\Psi}_{10} \\
D_1(\dot{\Phi}_{11}) = \beta_3 - \dot{\zeta}_4 - \frac{1}{2}\dot{\rho}_4 = i\dot{\Psi}_{11} & D_1(\dot{\Phi}_{12}) = \beta_4 + \dot{\zeta}_3 + \frac{1}{2}\dot{\rho}_3 = i\dot{\Psi}_{12}
\end{array} \tag{129}$$

For D_2,

$$\begin{array}{ll}
D_2(\dot{\Phi}_1) = \beta_2 + \frac{1}{2}\dot{\rho}_1 = i\dot{\Psi}_2 & D_2(\dot{\Phi}_2) = -\beta_1 + \frac{1}{2}\dot{\rho}_2 = -i\dot{\Psi}_1 \\
D_2(\dot{\Phi}_3) = \beta_4 - \frac{1}{2}\dot{\rho}_3 = i\dot{\Psi}_4 & D_2(\dot{\Phi}_4) = -\beta_3 - \frac{1}{2}\dot{\rho}_4 = -i\dot{\Psi}_3
\end{array} \tag{130}$$

$$D_2(\dot\Phi_5) = \beta_2 + \dot\zeta_1 - \frac{1}{2}\dot\rho_1 = i\dot\Psi_6 \qquad D_2(\dot\Phi_6) = -\beta_1 + \dot\zeta_2 - \frac{1}{2}\dot\rho_2 = -i\dot\Psi_5$$
$$D_2(\dot\Phi_7) = -\beta_4 + \dot\zeta_3 - \frac{1}{2}\dot\rho_3 = -i\dot\Psi_8 \qquad D_2(\dot\Phi_8) = \beta_3 + \dot\zeta_4 - \frac{1}{2}\dot\rho_4 = i\dot\Psi_7 \tag{131}$$

$$D_2(\dot\Phi_9) = -\beta_2 + \dot\zeta_1 + \frac{1}{2}\dot\rho_1 = -i\dot\Psi_{10} \qquad D_2(\dot\Phi_{10}) = \beta_1 + \dot\zeta_2 + \frac{1}{2}\dot\rho_2 = i\dot\Psi_9$$
$$D_2(\dot\Phi_{11}) = \beta_4 + \dot\zeta_3 + \frac{1}{2}\dot\rho_3 = i\dot\Psi_{12} \qquad D_2(\dot\Phi_{12}) = -\beta_3 + \dot\zeta_4 + \frac{1}{2}\dot\rho_4 = -i\dot\Psi_{11} \tag{132}$$

For D_3

$$D_3(\dot\Phi_1) = \beta_3 + \frac{1}{2}\dot\rho_4 = i\dot\Psi_3 \qquad D_3(\dot\Phi_2) = -\beta_4 + \frac{1}{2}\dot\rho_3 = -i\dot\Psi_4$$
$$D_3(\dot\Phi_3) = -\beta_1 + \frac{1}{2}\dot\rho_2 = -i\dot\Psi_1 \qquad D_3(\dot\Phi_4) = \beta_2 + \frac{1}{2}\dot\rho_1 = i\dot\Psi_2 \tag{133}$$

$$D_3(\dot\Phi_5) = -\beta_3 - \dot\zeta_4 + \frac{1}{2}\dot\rho_4 = -i\dot\Psi_7 \qquad D_3(\dot\Phi_6) = -\beta_4 + \dot\zeta_3 - \frac{1}{2}\dot\rho_3 = -i\dot\Psi_8$$
$$D_3(\dot\Phi_7) = \beta_1 - \dot\zeta_2 + \frac{1}{2}\dot\rho_2 = i\dot\Psi_5 \qquad D_3(\dot\Phi_8) = \beta_2 + \dot\zeta_1 - \frac{1}{2}\dot\rho_1 = i\dot\Psi_6 \tag{134}$$

$$D_3(\dot\Phi_9) = \beta_3 - \dot\zeta_4 - \frac{1}{2}\dot\rho_4 = i\dot\Psi_{11} \qquad D_3(\dot\Phi_{10}) = \beta_4 + \dot\zeta_3 + \frac{1}{2}\dot\rho_3 = i\dot\Psi_{12}$$
$$D_3(\dot\Phi_{11}) = -\beta_1 - \dot\zeta_2 - \frac{1}{2}\dot\rho_2 = -i\dot\Psi_9 \qquad D_3(\dot\Phi_{12}) = -\beta_2 + \dot\zeta_1 + \frac{1}{2}\dot\rho_1 = -i\dot\Psi_{10} \tag{135}$$

For D_4

$$D_4(\dot\Phi_1) = \beta_4 - \frac{1}{2}\dot\rho_3 = i\dot\Psi_4 \qquad D_4(\dot\Phi_2) = \beta_3 + \frac{1}{2}\dot\rho_4 = i\dot\Psi_3$$
$$D_4(\dot\Phi_3) = -\beta_2 - \frac{1}{2}\dot\rho_1 = -i\dot\Psi_2 \qquad D_4(\dot\Phi_4) = -\beta_1 + \frac{1}{2}\dot\rho_2 = -i\dot\Psi_1 \tag{136}$$

$$D_4(\dot\Phi_5) = -\beta_4 + \dot\zeta_3 - \frac{1}{2}\dot\rho_3 = -i\dot\Psi_8 \qquad D_4(\dot\Phi_6) = \beta_3 + \dot\zeta_4 - \frac{1}{2}\dot\rho_4 = i\dot\Psi_7$$
$$D_4(\dot\Phi_7) = -\beta_2 - \dot\zeta_1 + \frac{1}{2}\dot\rho_1 = -i\dot\Psi_6 \qquad D_4(\dot\Phi_8) = \beta_1 - \dot\zeta_2 + \frac{1}{2}\dot\rho_2 = i\dot\Psi_5 \tag{137}$$

$$D_4(\dot\Phi_9) = -\beta_4 - \dot\zeta_3 - \frac{1}{2}\dot\rho_3 = -i\dot\Psi_{12} \qquad D_4(\dot\Phi_{10}) = \beta_3 - \dot\zeta_4 - \frac{1}{2}\dot\rho_4 = i\dot\Psi_{11}$$
$$D_4(\dot\Phi_{11}) = -\beta_2 + \dot\zeta_1 + \frac{1}{2}\dot\rho_1 = -i\dot\Psi_{10} \qquad D_4(\dot\Phi_{12}) = \beta_1 + \dot\zeta_2 + \frac{1}{2}\dot\rho_2 = i\dot\Psi_9 \tag{138}$$

And for Fermionic fields,

$$D_1(i\dot{\Psi}_1) = i\dot{M} = i\ddot{\Phi}_1 \qquad D_1(i\dot{\Psi}_2) = -i\ddot{K} + i\dot{V}_0 = i\ddot{\Phi}_2$$
$$D_1(i\dot{\Psi}_3) = -i\ddot{L} + i\dot{U}_0 = i\ddot{\Phi}_3 \qquad D_1(i\dot{\Psi}_4) = i\dot{N} = i\ddot{\Phi}_4 \tag{139}$$

$$D_1(i\dot{\Psi}_5) = i\dot{U}_2 = i\ddot{\Phi}_5 \qquad D_1(i\dot{\Psi}_6) = -2i\ddot{K} + i\dot{V}_0 = i\ddot{\Phi}_6$$
$$D_1(i\dot{\Psi}_7) = -i\dot{U}_1 = i\ddot{\Phi}_7 \qquad D_1(i\dot{\Psi}_8) = i\dot{U}_3 = i\ddot{\Phi}_8 \tag{140}$$

$$D_1(i\dot{\Psi}_9) = -i\dot{V}_3 = i\ddot{\Phi}_9 \qquad D_1(i\dot{\Psi}_{10}) = -i\dot{V}_1 = i\ddot{\Phi}_{10}$$
$$D_1(i\dot{\Psi}_{11}) = -2i\ddot{L} + i\dot{U}_0 = i\ddot{\Phi}_{11} \qquad D_1(i\dot{\Psi}_{12}) = i\dot{V}_2 = i\ddot{\Phi}_{12} \tag{141}$$

$$D_2(i\dot{\Psi}_1) = i\ddot{K} - i\dot{V}_0 = -i\ddot{\Phi}_2 \qquad D_2(i\dot{\Psi}_2) = i\dot{M} = i\ddot{\Phi}_1$$
$$D_2(i\dot{\Psi}_3) = -i\dot{N} = -i\ddot{\Phi}_4 \qquad D_2(i\dot{\Psi}_4) = -i\ddot{L} + i\dot{U}_0 = i\ddot{\Phi}_3 \tag{142}$$

$$D_2(i\dot{\Psi}_5) = 2i\ddot{K} - i\dot{V}_0 = -i\ddot{\Phi}_6 \qquad D_2(i\dot{\Psi}_6) = i\dot{U}_2 = i\ddot{\Phi}_5$$
$$D_2(i\dot{\Psi}_7) = i\dot{U}_3 = i\ddot{\Phi}_8 \qquad D_2(i\dot{\Psi}_8) = i\dot{U}_1 = -i\ddot{\Phi}_7 \tag{143}$$

$$D_2(i\dot{\Psi}_9) = -i\dot{V}_1 = i\ddot{\Phi}_{10} \qquad D_2(i\dot{\Psi}_{10}) = i\dot{V}_3 = -i\ddot{\varphi}_9$$
$$D_2(i\dot{\Psi}_{11}) = -i\dot{V}_2 = -i\ddot{\varphi}_{12} \qquad D_2(i\dot{\Psi}_{12}) = -2i\ddot{L} + i\dot{U}_0 = i\ddot{\Phi}_{11} \tag{144}$$

$$D_3(i\dot{\Psi}_1) = i\ddot{L} - i\dot{U}_0 = -i\ddot{\Phi}_3 \qquad D_3(i\dot{\Psi}_2) = i\dot{N} = i\ddot{\Phi}_4$$
$$D_3(i\dot{\Psi}_3) = i\dot{M} = i\ddot{\Phi}_1 \qquad D_3(i\dot{\Psi}_4) = i\ddot{K} - i\dot{V}_0 = -i\ddot{\Phi}_2 \tag{145}$$

$$D_3(i\dot{\Psi}_5) = -i\dot{U}_1 = i\ddot{\Phi}_7 \qquad D_3(i\dot{\Psi}_6) = i\dot{U}_3 = i\ddot{\Phi}_8$$
$$D_3(i\dot{\Psi}_7) = -i\dot{U}_2 = -i\ddot{\Phi}_5 \qquad D_3(i\dot{\Psi}_8) = 2i\ddot{K} - i\dot{V}_0 = -i\ddot{\Phi}_6 \tag{146}$$

$$D_3(i\dot{\Psi}_9) = 2i\ddot{L} - i\dot{U}_0 = -i\ddot{\Phi}_{11} \qquad D_3(i\dot{\Psi}_{10}) = -i\dot{V}_2 = -i\ddot{\Phi}_{12}$$
$$D_3(i\dot{\Psi}_{11}) = -i\dot{V}_3 = i\ddot{\Phi}_9 \qquad D_3(i\dot{\Psi}_{12}) = -i\dot{V}_1 = i\ddot{\Phi}_{10} \tag{147}$$

$$D_4(i\dot{\Psi}_1) = -i\dot{N} = -i\ddot{\Phi}_4 \qquad D_4(i\dot{\Psi}_2) = i\ddot{L} - i\dot{U}_0 = -i\ddot{\Phi}_3$$
$$D_4(i\dot{\Psi}_3) = -i\ddot{K} + i\dot{V}_0 = i\ddot{\Phi}_2 \qquad D_4(i\dot{\Psi}_4) = i\dot{M} = i\ddot{\Phi}_1 \tag{148}$$

$$D_4(i\dot{\Psi}_5) = i\dot{U}_3 = i\ddot{\Phi}_8 \qquad D_4(i\dot{\Psi}_6) = i\dot{U}_1 = -i\ddot{\Phi}_7$$
$$D_4(i\dot{\Psi}_7) = -2i\ddot{K} + i\dot{V}_0 = i\ddot{\Phi}_6 \qquad D_4(i\dot{\Psi}_8) = -i\dot{U}_2 = -i\ddot{\Phi}_5 \tag{149}$$

$$D_4(i\dot{\Psi}_9) = i\dot{V}_2 = i\ddot{\Phi}_{12} \qquad D_4(i\dot{\Psi}_{10}) = 2i\ddot{L} - i\dot{U}_0 = -i\ddot{\Phi}_{11}$$
$$D_4(i\dot{\Psi}_{11}) = -i\dot{V}_1 = i\ddot{\Phi}_{10} \qquad D_4(i\dot{\Psi}_{12}) = i\dot{V}_3 = -i\ddot{\Phi}_9 \tag{150}$$

3.4 L and R Matrices

The L-matrix can be written as,

$$L_1 = \left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \tag{151}$$

$$L_2 = \left[\begin{array}{cccc|cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right] \tag{152}$$

$$L_3 = \left[\begin{array}{cccc|cccc|cccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{array}\right] \tag{153}$$

$$
L_4 = \left[\begin{array}{cccc|cccc|cccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right] \qquad (154)
$$

In the same logic, the R-matrix can be written as,

$$
R_1 = \left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \qquad (155)
$$

$$
R_2 = \left[\begin{array}{cccc|cccc|cccc}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] \qquad (156)
$$

$$
R_3 = \left[\begin{array}{cccc|cccc|cccc}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right] \tag{157}
$$

$$
R_4 = \left[\begin{array}{cccc|cccc|cccc}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{array}\right] \tag{158}
$$

So we can check $L_i = {R_i}^T$, which satisfy the garden algebra.

3.5 Adinkras

We can draw the three separate Adinkra for the CLS.

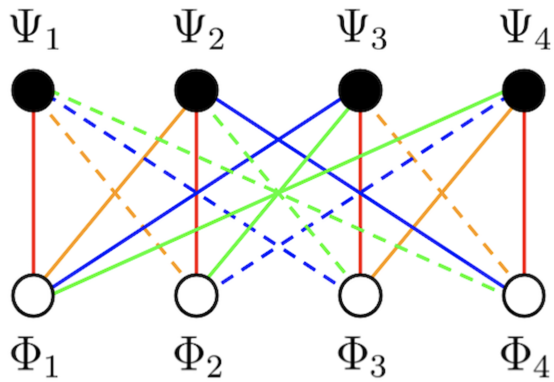

Figure 1: From field number 1 to 4

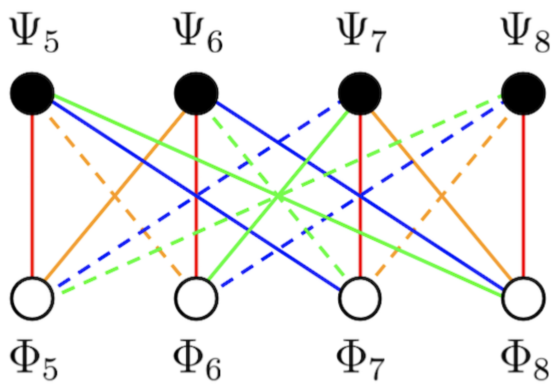

Figure 2: From field number 5 to 8

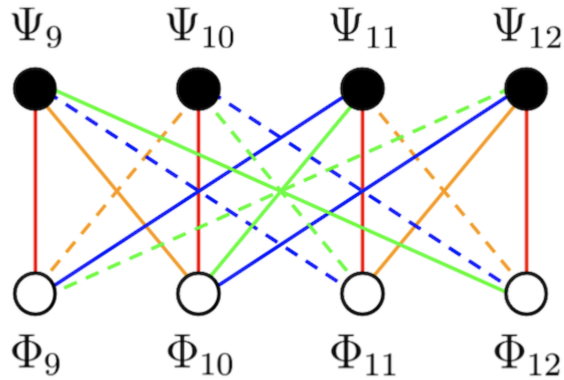

Figure 3: From field number 9 to 12

Here, red, yellow, blue, green indicate D_1, D_2, D_3, D_4 respectively. A dotted line represents a negative relationship and a filled line represents a positive relationship between the two fields.

4 Reference

[1] Aleksander J. Cianciara, S. James Gates Yangrui Hu and Konstantinos Koutrolikos, Lecture note on 2022 SSTPRS

[2] Gates, S. James and Hu, Yangrui and Mak, S. -N. Hazel, 2021, On 1D, N = 4 Supersymmetric SYK-Type Models (II),

https://doi.org/10.48550/arxiv.2110.15562

[3] S.-N. Hazel Mak, and Kory Stiffler, 4D, N = 1 Matter Gravitino Genomics, 2019,10.3390/sym11020217

https://www.mdpi.com/2073-8994/11/2/217